Aquarius: Exact Reading and Writing in a Closed Circulation

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Abstract

One closed circulation of water runs through this paper, from the heat that turns it to what a single return reads and writes, and one constant of the vessel's shape governs both. Inside a closed ellipsoid of equatorial radius R, the circulation with streamfunction $\Psi(u, w) = u(1-u-w^2)$ solves the steady incompressible Euler and Navier–Stokes momentum equations at every viscosity $\mu \geq 0$, while its midplane pressure takes its minimum on the ring $r_* = R/\sqrt{2}$ that splits the midplane into equal areas. At uniform susceptibility a static magnetic field sets the interior pressure, and for axisymmetric profiles crossed gradients of thermal susceptibility and magnetic energy drive vorticity directly along the circulation's own axis. Twelve coupled vessels on a ring split as $\mathbb{Z}/12\mathbb{Z} \cong \mathsf{Trit} \times \mathbb{Z}/4\mathbb{Z}$, the table of modes and elements of the twelve signs, and wherever three branches meet a junction holds the carry trit $t(a, b) = a + b$, which one scalar bias reads as NOR or NAND. The field sets that bias, so one network of junctions with fixed wiring computes every Boolean function of its inputs, its program held in the field. All of it composes into one factorization, $f_t = D \circ \Phi_t \circ E = \bar{D} \circ \bar{\Phi}_t \circ \bar{E}$, under which the main theorems of eleven of the author's sixteen earlier papers fall as instances of Theorem A0 and its bridge laws. On every interior stream surface ($0 < a < \frac{1}{2}$) the shape constant is $K > 0$, determined by the aspect ratio and the shell alone: the return angle reads a relation among three phases that no pair of modes carries, a small torsion turns independent phases into a relation among three modes written in the shifted amplitudes, and at constant inertia $K = 0$ and both leading channels go silent. The theorems are machine-checked in Lean 4 with Mathlib, and their analytic parts, among them the elliptic form of the period in Theorem A3 and the uniform remainder bounds of Theorem A12(a), are proved in the text.

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1 The return and its contraction

> *We must think of it as a river flowing up and down in a circle and made up partly of air, partly of water.* [...] *So if 'Oceanus' had some secret meaning in early writers, perhaps they may have meant this river that flows in a circle about the earth.*
>
> Aristotle, *Meteorologica* I.9 [2]

Aristotle is describing rain. Heat draws the water up as vapour and cold brings it down again, and in this circle, he says, the order of the sequence always remains the same; in the Preface to the *Phenomenology*, Hegel calls the true the whole, the circle that presupposes its end as its goal and has its end for its beginning [17]. Aristotle thought the early writers had already named that circle, and that their Oceanus was this river.

Aquarius is another old name for it. It belongs to the signs of air, which Aristotle held to be hot and moist, "a sort of aqueous vapour" [3], while its figure carries water, making the sign air carrying water, the same river held in one image. Its oldest form is the Babylonian GU.LA, the Great One, the god Ea holding a vase that overflows [51], and Ea was the god of the fresh water beneath the earth and of wisdom [5], the water and the knowledge in one figure from the start. Its glyph is two waves, one above the other, and the age that carries its name is read as the one in which that water is poured out for everyone.

The twelve signs of the zodiac form a table of three modes and four elements in which no two signs share both, and that table is the residue split $\mathbb{Z}/12\mathbb{Z} \cong \mathsf{Trit} \times \mathbb{Z}/4\mathbb{Z}$ of Theorem A0(d). Counting Aries as zero, Aquarius stands at ten, fixed and air, the pair $(1, 2)$, and $4 \cdot 1 + 9 \cdot 2 = 22 \equiv 10 \pmod{12}$ gives it back. The same split carries the ring of twelve vessels in Section 4 to the trit of Section 5.

Sixteen papers before this one kept returning to the same question: when a large carrier evolves under a simple law, what does an observation keep of it? Each time, a small boundary appeared through which everything downstream factors. An ordered decision keeps one of three states [40], thirty-two additions sharing an input keep at most thirty-three carry states across a cut [34], and every preimage of a cyclic Boolean–modular map is fixed by its first three bits [27]. Theorem A0 states that contraction once for any flow, and Appendix A gathers the main theorems of eleven of those sixteen papers directly under it and its bridge laws, alongside five that stand in an explicit relation to it. Now we put the same question to a moving fluid: what does a closed circulation keep?

Write every computation in this paper as one transformation,

$$f_t = D \circ \Phi_t \circ E : \mathcal{X} \xrightarrow{E} \mathcal{G} \xrightarrow{\Phi_t} \mathcal{G} \xrightarrow{D} \mathcal{Y}. \tag{1}$$

Here $\mathcal{G}$ is the state space of a geometry, $(\Phi_t)_{t \in \mathbb{T}}$ is its evolution, $E : \mathcal{X} \to \mathcal{G}$ prepares an initial, boundary or spectral configuration from an input in $\mathcal{X}$, and $D : \mathcal{G} \to \mathcal{Y}$ observes it. The geometry carries the state, and its evolution is the computation. A finite state machine, a

carry trit and a one-bit gate are all contractions D of that evolved geometry, some exact and some coarse, so what the computation carries forward is whatever D can tell apart along the flow; the distinction itself is the primitive. Reinhold set one act of distinction at the root in his principle of consciousness, where the representation is distinguished by the subject from subject and object and related to both [50].

Quotienting a carrier by what an observation can tell apart runs across four traditions: it gives Nerode's minimal automaton [47], the observable part of a linear system in Kalman's canonical decomposition [20], a lumpable Markov chain [23], and symmetry reduction for a flow [43]. Kant set the thing as it is in itself apart from the thing as it appears; only the appearance, shaped by the observer's own forms, is received [21]. Theorem A0 formulates that quotient once for a flow semigroup and its iterates, setting up the exact contractions used through every section that follows.

Definition 1.1 (Flow semigroup and equivariant contraction)**.** Let $\mathcal{G}$ be a set. A one-parameter family $\Phi : \mathbb{R} \to (\mathcal{G} \to \mathcal{G})$ is a *flow semigroup* when $\Phi_0(g) = g$ and $\Phi_{t+s}(g) = \Phi_t(\Phi_s(g))$ for all $t, s \in \mathbb{R}$ and $g \in \mathcal{G}$. For a single-step map $F : \mathcal{G} \to \mathcal{G}$, its discrete iterates are $F^0(g) = g$ and $F^{n+1}(g) = F(F^n(g))$ $(n \in \mathbb{N})$. A surjection $\pi : \mathcal{G} \to \mathcal{Q}$ has *F-invariant kernel* when $\pi(g_1) = \pi(g_2)$ implies $\pi(F(g_1)) = \pi(F(g_2))$, and is *flow-equivariant* with respect to $\bar{\Phi} : \mathbb{R} \to (\mathcal{Q} \to \mathcal{Q})$ when $\pi \circ \Phi_t = \bar{\Phi}_t \circ \pi$ for every $t \in \mathbb{R}$.

Following [40, Section 2], a family of observations $(O_i : \mathcal{G} \to V_i)_{i \in I}$ induces the observation equivalence $g_1 \sim_{\mathcal{O}} g_2 \iff \forall i,\, O_i(g_1) = O_i(g_2)$. A map $b : \mathcal{G} \to B$ is a *retained boundary* when $b(g_1) = b(g_2) \implies g_1 \sim_{\mathcal{O}} g_2$, and *exact* when the converse holds as well.

Theorem A0 (The contraction of a flow)**.** *Let $\mathcal{G}$ be a state space, $E : \mathcal{X} \to \mathcal{G}$, $D : \mathcal{G} \to \mathcal{Y}$, and $\pi : \mathcal{G} \to \mathcal{Q}$ a surjective map.*

(a) For any map $F : \mathcal{G} \to \mathcal{G}$, there exists a reduced map $\bar{F} : \mathcal{Q} \to \mathcal{Q}$ satisfying $\pi \circ F = \bar{F} \circ \pi$ if and only if π has F-invariant kernel. When $\bar{F}$ exists, it is unique and satisfies $\pi \circ F^n = \bar{F}^n \circ \pi$ for every $n \in \mathbb{N}$.

(b) If Φ is a flow semigroup on $\mathcal{G}$ and $\pi \circ \Phi_t = \bar{\Phi}_t \circ \pi$ for every $t \in \mathbb{R}$, then $\bar{\Phi}$ is a flow semigroup on $\mathcal{Q}$. Whenever $D = \bar{D} \circ \pi$ for some $\bar{D} : \mathcal{Q} \to \mathcal{Y}$, setting $\bar{E} = \pi \circ E$ contracts (1) *at every horizon $t \in \mathbb{R}$:*

$$f_t = D \circ \Phi_t \circ E = \bar{D} \circ \bar{\Phi}_t \circ \bar{E}. \tag{2}$$

(c) For any observation family $\mathcal{O}$ on $\mathcal{G}$, every finite retained boundary $b : \mathcal{G} \to B$ bounds the quotient by $|\mathcal{G}/\!\sim_{\mathcal{O}}| \le |B|$, and every surjective exact boundary $b : \mathcal{G} \to B$ determines a unique bijection $e : B \xrightarrow{\sim} \mathcal{G}/\!\sim_{\mathcal{O}}$ with $e \circ b = [\,\cdot\,]_{\mathcal{O}}$.

(d) On the twelve-tick cyclic dial $\mathbb{Z}/12\mathbb{Z}$, Mordell's identity

$$x \equiv 4\,(x \bmod 3) - 3\,(x \bmod 4) \pmod{12} \qquad (x \in \mathbb{Z}) \tag{3}$$

induces the bijection $\mu_{12} : \mathrm{Fin}\,12 \xrightarrow{\sim} \mathrm{Fin}\,3 \times \mathrm{Fin}\,4$, $x \mapsto (x \bmod 3,\, x \bmod 4)$, *with inverse* $(r_3, r_4) \mapsto (4r_3 + 9r_4) \bmod 12$. *For any* $n \geq 1$, $a \in \mathbb{Z}$, *and angular reading* $\vartheta \in \mathbb{R}$ *with* $|\vartheta - a/n| < 1/(2n)$, *the rounded gauge* $\lfloor n\vartheta + \frac{1}{2} \rfloor$ *recovers* a; *reading* $\mathbb{Z}/12\mathbb{Z}$ *through* $(\mathrm{Fin}\,3, \mathrm{Fin}\,4)$ *therefore tolerates uniform phase error* $1/8$ *per channel, three times the* $1/24$ *of the undivided twelve-state dial.*

Proof. (a) If $\pi \circ F = \bar{F} \circ \pi$ and $\pi(g_1) = \pi(g_2)$, then $\pi(F(g_1)) = \bar{F}(\pi(g_1)) = \bar{F}(\pi(g_2)) = \pi(F(g_2))$. Conversely, assume $\ker \pi$ is F-invariant. Because π is surjective, each $q \in \mathcal{Q}$ has a chosen preimage $s(q) \in \mathcal{G}$ with $\pi(s(q)) = q$. Put $\bar{F}(q) = \pi(F(s(q)))$. For any $g \in \mathcal{G}$, $\pi(s(\pi(g))) = \pi(g)$, so kernel invariance gives $\pi(F(g)) = \pi(F(s(\pi(g)))) = \bar{F}(\pi(g))$. If $\bar{F}_1 \circ \pi = \bar{F}_2 \circ \pi$, surjectivity of π gives $\bar{F}_1 = \bar{F}_2$. Induction on n gives $\pi(F^0(g)) = \pi(g) = \bar{F}^0(\pi(g))$ and $\pi(F^{n+1}(g)) = \bar{F}(\pi(F^n(g))) = \bar{F}(\bar{F}^n(\pi(g))) = \bar{F}^{n+1}(\pi(g))$.

(b) Given $q = \pi(g)$, $\bar{\Phi}_0(q) = \pi(\Phi_0(g)) = \pi(g) = q$, and for all $t, s \in \mathbb{R}$,

$$\bar{\Phi}_{t+s}(q) = \pi(\Phi_{t+s}(g)) = \pi(\Phi_t(\Phi_s(g))) = \bar{\Phi}_t(\pi(\Phi_s(g))) = \bar{\Phi}_t(\bar{\Phi}_s(q)).$$

When $D = \bar{D} \circ \pi$, $(D \circ \Phi_t \circ E)(x) = \bar{D}(\pi(\Phi_t(E(x)))) = \bar{D}(\bar{\Phi}_t(\pi(E(x)))) = (\bar{D} \circ \bar{\Phi}_t \circ \bar{E})(x)$.

(c) Sending $b(g) \in b(\mathcal{G})$ to $[g]_{\mathcal{O}}$ is well defined because b is a retained boundary, and surjective because $[\,\cdot\,]_{\mathcal{O}}$ is surjective; hence $|\mathcal{G}/{\sim_{\mathcal{O}}}| \leq |b(\mathcal{G})| \leq |B|$. When b is exact and surjective, this map is defined on all of B and is injective because $[g_1]_{\mathcal{O}} = [g_2]_{\mathcal{O}} \implies g_1 \sim_{\mathcal{O}} g_2 \implies b(g_1) = b(g_2)$; uniqueness follows from surjectivity of b.

(d) Write $x \in \mathbb{Z}$: since $3 \mid 12$ and $4 \mid 12$, $4(x \bmod 3) - 3(x \bmod 4)$ is congruent to $4x - 3x = x$ modulo 3 (as $4x \equiv x$ and $-3(x \bmod 4) \equiv 0$) and modulo 4 (as $4(x \bmod 3) \equiv 0$ and $-3x \equiv x$), hence modulo $\mathrm{lcm}(3, 4) = 12$. Since $-3 \equiv 9 \pmod{12}$, $(4r_3 + 9r_4) \bmod 12$ supplies the exact inverse on $\mathrm{Fin}\,3 \times \mathrm{Fin}\,4$. Finally, $|\vartheta - a/n| < 1/(2n)$ multiplies by $n > 0$ to $-1/2 < n\vartheta - a < 1/2$, or $a < n\vartheta + 1/2 < a + 1$, so $\lfloor n\vartheta + 1/2 \rfloor = a$. With $n = 3$ and $n = 4$ the two channels decode below $1/6$ and $1/8$, so every error below $1/8$ is read exactly on both. □

Under μ_{12} the twelve signs, numbered from Aries at 0 to Pisces at 11, split directly into their modes and elements: cardinal, fixed and mutable are $x \bmod 3 = 0, 1, 2$, and fire, earth, air and water are $x \bmod 4 = 0, 1, 2, 3$. In binary, $x \bmod 4$ is Aristotle's pair of qualities [3]. Its low bit alternates hot and cold from sign to sign, and its high bit, flipped by the carry out of the low bit, alternates dry and moist every second sign.

The chain

Sections 2 to 8 trace one chain,

$$\text{thermal energy} \longrightarrow \text{resonance} \longrightarrow \text{geometry} \longrightarrow \text{state} \longrightarrow \text{information}, \tag{4}$$

and each arrow carries its full hypotheses into the theorem that proves it. Section 2 builds the vessel and its circulation; Section 3 turns the water with heat and sets the pressure with the field; Section 4 puts twelve vessels on a ring; Section 5 finds the trit where three branches

meet; Section 6 composes the arrows into one theorem; Section 7 follows a single turn of the water to the constant that reads and writes; and Section 8 writes the program into the field, so one fixed network computes every function.

2 The vessel

Let $R, H, J_0, \rho > 0$, put $U = J_0/\rho$, and define the closed ellipsoid

$$\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 : \frac{x^2 + y^2}{R^2} + \frac{z^2}{H^2} \le 1 \right\}. \tag{5}$$

Write $X = x/R$, $Y = y/R$, $w = z/H$, $r = \sqrt{x^2 + y^2}$, $u = r^2/R^2 = X^2 + Y^2$, and $B(x, y, z) = u + w^2$, so that $\Omega = \{B \le 1\}$ and $\partial\Omega = \{B = 1\}$. Let $\mu_z(x, y, z) = (x, y, -z)$ reflect across the equatorial plane $z = 0$. Extending Theorem W5 of [41] to $\mathbb{R}^3$, every scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ splits uniquely into a mirror-even *drive* $f^+(p) = \frac{1}{2}(f(p) + f(\mu_z(p)))$ and a mirror-odd *torsion* $f^-(p) = \frac{1}{2}(f(p) - f(\mu_z(p)))$, vanishing across the midplane as $f^-(x, y, 0) = 0$. Under μ_z the upper and lower sheets trade places, the Emerald Tablet's "as above, so below" [53].

Theorem A1 (The circulation and its two sheets)**.** *Define the normalized streamfunction* $\Psi(u, w) = u(1 - u - w^2)$ *and Stokes streamfunction* $\mathcal{S}(r, z) = \frac{1}{2} J_0 R^2 \Psi(u, w)$*. The resulting Cartesian mass-flux field*

$$\mathbf{J}(x, y, z) = J_0 \left(\frac{xz}{H^2},\ \frac{yz}{H^2},\ 1 - \frac{2(x^2 + y^2)}{R^2} - \frac{z^2}{H^2} \right) \tag{6}$$

satisfies $\nabla \cdot \mathbf{J} = 0$ *on all of* $\mathbb{R}^3$*,* $\nabla B \cdot \mathbf{J} = \frac{2J_0}{H} w(1 - B)$ *so that outward normal flux vanishes everywhere on* $\partial\Omega$*, and* $\nabla\Psi \cdot \mathbf{J} = 0$*. Under the reflection* $\mu_z(x, y, z) = (x, y, -z)$*, the field obeys* $\mathbf{J}(\mu_z(p)) = (-J_x(p), -J_y(p), J_z(p))$*.*

Proof. For $r > 0$, $-r^{-1}\partial_z \mathcal{S} = J_0 rz/H^2$ and $r^{-1}\partial_r \mathcal{S} = J_0(1 - 2u - w^2)$, whose Cartesian lift is the polynomial vector field (6) on all of $\mathbb{R}^3$. Direct differentiation gives

$$\partial_x J_x + \partial_y J_y + \partial_z J_z = \frac{J_0 z}{H^2} + \frac{J_0 z}{H^2} - \frac{2J_0 z}{H^2} = 0.$$

Since $\nabla B = (2x/R^2, 2y/R^2, 2z/H^2)$,

$$\nabla B \cdot \mathbf{J} = J_0 \left(\frac{2(x^2 + y^2)z}{R^2 H^2} + \frac{2z}{H^2}(1 - 2u - w^2) \right) = \frac{2J_0}{H} w(1 - u - w^2) = \frac{2J_0}{H} w(1 - B),$$

which vanishes when $B = 1$. Using $\partial_u \Psi = 1 - 2u - w^2$ and $\partial_w \Psi = -2uw$,

$$\nabla\Psi \cdot \mathbf{J} = (1 - 2u - w^2) \left(\frac{2x}{R^2} J_x + \frac{2y}{R^2} J_y \right) - 2uw\, \frac{1}{H} J_z = \frac{2J_0}{H} uw\big((1 - 2u - w^2) - (1 - 2u - w^2)\big) = 0.$$

Replacing z by $-z$ negates xz and yz while preserving z^2, yielding the reflection law. □

Theorem A2 (The ring of equal areas)**.** *Across the equatorial disc $\Omega \cap \{z = 0\}$, the normalized cumulative vertical transfer inside squared normalized radius $u \in [0,1]$ is $C(u) = \int_0^u (1-2s)\,\mathrm{d}s = u - u^2$. The vertical flux $J_z(x,y,0) = J_0(1-2u)$ vanishes only at $u_* = 1/2$ ($r_* = R/\sqrt{2}$), dividing $\Omega \cap \{z = 0\}$ into an inner disc and an outer annulus of equal area $\pi R^2/2$ carrying balanced upward and downward transfers*

$$Q_+ = \frac{\pi R^2 J_0}{4}, \qquad Q_- = -\frac{\pi R^2 J_0}{4}, \qquad Q_+ + Q_- = 0. \tag{7}$$

Among all vector fields on Ω of the form $\tilde{\mathbf{J}}(x,y,z) = J_0\big(xz/H^2,\, yz/H^2,\, f(u) - w^2\big)$ with arbitrary $f : [0,1] \to \mathbb{R}$, boundary tangency $\nabla B \cdot \tilde{\mathbf{J}} = 0$ on $\partial\Omega = \{u + w^2 = 1\}$ forces $f(u) = 1 - 2u$ for every $u \in [0,1]$.

Proof. In polar coordinates on $z = 0$, $2\pi r\,\mathrm{d}r = \pi R^2\,\mathrm{d}u$, so the cumulative flux through radius $r = R\sqrt{u}$ is $\pi R^2 J_0 \int_0^u (1-2s)\,\mathrm{d}s = \pi R^2 J_0 (u - u^2)$. Clearly $1 - 2u = 0 \iff u = 1/2$, with $C(1/2) - C(0) = 1/4$ and $C(1) - C(1/2) = -1/4$, giving (7); the two regions occupy $\Delta u = 1/2$, hence area $\pi R^2/2$ each. For uniqueness, $\nabla B \cdot \tilde{\mathbf{J}} = \frac{2J_0}{H} w\big(u + f(u) - w^2\big)$. Requiring $u + f(u) - w^2 = 0$ whenever $u + w^2 = 1$ (or by continuity from $w = \sqrt{1-u} > 0$ on $u \in [0,1)$ to $u = 1$), choose $w = \sqrt{1-u}$ for any $u \in [0,1]$: then $w^2 = 1 - u$, so $u + f(u) - (1-u) = 0$, whence $f(u) = 1 - 2u$. □

In the continuum, (7) is the seam dipole of [41, Theorem W6]. The inner disc $u < 1/2$ forms the upward source $+Q_+$, and the outer annulus $1/2 < u \le 1$ carries its return field, the sink $-Q_+$.

Theorem A3 (Each turn contracts to a circle)**.** *In normalized time $\tau = Ut/H$, interior trajectories of $\mathbf{v} = \mathbf{J}/\rho$ where $\Psi > 0$ follow the meridional system $\dot{u} = 2uw$, $\dot{w} = 1 - 2u - w^2$ ($\dot{} = \mathrm{d}/\mathrm{d}\tau$), which contracts under $\Pi_{\mathrm{circ}}(u,w) = (u,\, w\sqrt{u}) = (u,v)$ onto concentric circles*

$$\left(u - \frac{1}{2}\right)^2 + v^2 = a^2, \qquad a^2 = \frac{1}{4} - \Psi, \qquad 0 < a < \frac{1}{2}. \tag{8}$$

Each circle is parametrized inside $\Omega^\circ = \{u + w^2 < 1\}$ by $u(\theta) = \frac{1}{2} - a\cos\theta$, $v(\theta) = a\sin\theta$, $w(\theta) = a\sin\theta/\sqrt{\frac{1}{2} - a\cos\theta}$, with phase velocity $\dot{\theta} = 2\sqrt{\frac{1}{2} - a\cos\theta} > 0$ and normalized period

$$P(a) = \int_0^{2\pi} \frac{\mathrm{d}\theta}{2\sqrt{1/2 - a\cos\theta}} = \frac{2}{\sqrt{1/2 + a}}\, K\!\left(\frac{2a}{1/2 + a}\right), \tag{9}$$

where $K(m) = \int_0^{\pi/2} (1 - m\sin^2\varphi)^{-1/2}\,\mathrm{d}\varphi$ ($0 \le m < 1$). The physical period is $HP(a)/U$.

Proof. For $0 \le a < 1/2$, $u(\theta) = \frac{1}{2} - a\cos\theta \ge \frac{1}{2} - a > 0$. Substituting $v = w\sqrt{u}$ into $\Psi = u(1 - u - w^2) = u - u^2 - v^2$ completes the square to $(u - 1/2)^2 + v^2 = 1/4 - \Psi = a^2$, and

$u + w^2 = 1 - \Psi/u = 1 - (1/4 - a^2)/u < 1$, proving strict interior enclosure. Differentiating $v = w\sqrt{u}$ along $\dot{u} = 2uw = 2\sqrt{u}\,v$ and $\dot{w} = 1 - 2u - w^2$ gives

$$\dot{v} = \dot{w}\sqrt{u} + w\frac{\dot{u}}{2\sqrt{u}} = (1 - 2u - w^2)\sqrt{u} + w^2\sqrt{u} = \sqrt{u}(1 - 2u).$$

Along $(u(\theta), v(\theta))$, the chain rule with $\dot{\theta} = 2\sqrt{u(\theta)}$ gives $\dot{u} = (a\sin\theta)(2\sqrt{u}) = 2\sqrt{u}\,v$ and $\dot{v} = (a\cos\theta)(2\sqrt{u}) = 2\sqrt{u}(\frac{1}{2} - u) = \sqrt{u}(1 - 2u)$, matching the meridional ODE identically. Since $\dot{\theta} \geq 2\sqrt{1/2 - a} > 0$, one revolution takes normalized time $P(a) = \int_0^{2\pi}(2\sqrt{1/2 - a\cos\theta})^{-1}\,\mathrm{d}\theta$. By symmetry on $[0, \pi]$ and $[-\pi, 0]$, substituting $\theta = \pi - 2\varphi$ ($\varphi \in [0, \pi/2]$) gives $1/2 - a\cos\theta = 1/2 + a\cos(2\varphi) = (1/2 + a)(1 - m\sin^2\varphi)$ with $m = 2a/(1/2 + a) \in (0, 1)$, yielding (9). □

Hill introduced a spherical Euler vortex in 1894 [18], recorded in Lamb [25], and O'Brien generalized it to viscous spheroids [48], surveyed by Wang [59]. The velocity of Theorem A1 realizes that family at arbitrary aspect ratio $\eta = H/R$ and every viscosity $\mu \geq 0$, with the wall a stream surface along which the water slides: the velocity never depends on μ, and viscosity enters the pressure only as a term linear in height.

Theorem A4 (Every ellipsoid, every viscosity)**.** *For all $R, H, \rho > 0$, all $U \in \mathbb{R}$, and every dynamic viscosity $\mu \geq 0$, the velocity field $\mathbf{v} = \mathbf{J}/\rho = \big(Uxz/H^2,\, Uyz/H^2,\, U(1 - 2u - w^2)\big)$ has convective acceleration*

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = \left(\frac{U^2}{H^2}x(1 - 2u),\ \frac{U^2}{H^2}y(1 - 2u),\ -\frac{2U^2}{H^2}z(1 - w^2)\right) \tag{10}$$

and solves the steady three-dimensional incompressible Navier–Stokes momentum equation

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p_\mu + \mu\Delta\mathbf{v} \tag{11}$$

throughout Ω with interior pressure

$$p_\mu(x, y, z) = p_0 - \frac{\rho U^2 R^2}{2H^2}(u - u^2) + \rho U^2\left(w^2 - \frac{w^4}{2}\right) - 2\mu U\left(\frac{4}{R^2} + \frac{1}{H^2}\right)z. \tag{12}$$

When $U \neq 0$, the midplane pressure $p_\mu(x, y, 0) = p_0 - \frac{\rho U^2 R^2}{2H^2}C(u)$ takes its strict global minimum $p_{\min} = p_0 - \frac{\rho U^2 R^2}{8H^2}$ on the equal-area split circle $u_* = 1/2$ ($r_* = R/\sqrt{2}$). Its vorticity $\boldsymbol{\omega}_{\mathrm{vort}} = \nabla\times\mathbf{v} = U\big(\frac{4}{R^2} + \frac{1}{H^2}\big)(-y, x, 0)$ is harmonic, $\Delta\boldsymbol{\omega}_{\mathrm{vort}} = \mathbf{0}$, and obeys $(\mathbf{v}\cdot\nabla)\boldsymbol{\omega}_{\mathrm{vort}} = (\boldsymbol{\omega}_{\mathrm{vort}}\cdot\nabla)\mathbf{v}$.

Proof. Computing the velocity Jacobian gives $\partial_x v_x = \partial_y v_y = Uz/H^2$, $\partial_z v_x = Ux/H^2$, $\partial_z v_y = Uy/H^2$, $\partial_x v_z = -4Ux/R^2$, $\partial_y v_z = -4Uy/R^2$, $\partial_z v_z = -2Uz/H^2$, and $\partial_y v_x = \partial_x v_y = 0$. Thus

$$\begin{aligned}
(\mathbf{v}\cdot\nabla)v_x &= \frac{Uxz}{H^2}\frac{Uz}{H^2} + U(1 - 2u - w^2)\frac{Ux}{H^2} = \frac{U^2}{H^2}x(1 - 2u),\\
(\mathbf{v}\cdot\nabla)v_z &= \frac{Uz}{H^2}(x\partial_x v_z + y\partial_y v_z) - \frac{2U^2 z}{H^2}(1 - 2u - w^2)\\
&= -\frac{2U^2}{H^2}z(2u + 1 - 2u - w^2) = -\frac{2U^2}{H^2}z(1 - w^2),
\end{aligned}$$

and symmetrically for $(\mathbf{v}\cdot\nabla)v_y$, establishing (10). Second partial derivatives of v_x, v_y all vanish, while $\Delta v_z = -4U/R^2 - 4U/R^2 - 2U/H^2 = -2U(4/R^2 + 1/H^2)$. Differentiating (12) using $\nabla u = (2x/R^2, 2y/R^2, 0)$ and $\partial_z w = 1/H$ yields

$$-\nabla p_\mu + \mu\Delta\mathbf{v} = \left(\frac{\rho U^2}{H^2}x(1-2u),\ \frac{\rho U^2}{H^2}y(1-2u),\ -\frac{2\rho U^2}{H^2}z(1-w^2)\right) = \rho(\mathbf{v}\cdot\nabla)\mathbf{v},$$

proving (11). At $z = 0$,

$$p_\mu(x,y,0) - \left(p_0 - \frac{\rho U^2 R^2}{8H^2}\right) = \frac{\rho U^2 R^2}{2H^2}\left(\frac{1}{4} - u + u^2\right) = \frac{\rho U^2 R^2}{2H^2}\left(u - \frac{1}{2}\right)^2 > 0$$

for every $u \neq 1/2$ when $\rho, R, H > 0$ and $U \neq 0$. Finally, $\nabla\times\mathbf{v} = (\partial_y v_z - \partial_z v_y,\ \partial_z v_x - \partial_x v_z,\ 0) = U(4/R^2 + 1/H^2)(-y, x, 0)$. Because $\boldsymbol{\omega}_{\text{vort}}$ is linear in (x, y) and independent of z, $\Delta\boldsymbol{\omega}_{\text{vort}} = \mathbf{0}$, while $(\mathbf{v}\cdot\nabla)\boldsymbol{\omega}_{\text{vort}} = U(4/R^2+1/H^2)(-v_y,\ v_x,\ 0) = \frac{U^2 z}{H^2}(4/R^2+1/H^2)(-y, x, 0) = (\boldsymbol{\omega}_{\text{vort}}\cdot\nabla)\mathbf{v}$, since $\omega_x\partial_x v_z + \omega_y\partial_y v_z \propto (-y)(-x) + x(-y) = 0$. □

With a swirl constant in height, the motion on every interior stream surface conjugates to a Kronecker flow on a torus, and the water closes back on itself if and only if the swirl rate and meridional period are commensurate, $\omega P(a)/2\pi \in \mathbb{Q}$.

Theorem A5 (Two clocks on a torus)**.** *For any constant normalized swirl rate $\omega \in \mathbb{R}$, the normalized vector field*

$$\mathbf{F}_\omega(X, Y, w) = \big(Xw + \omega Y,\ Yw - \omega X,\ 1 - 2(X^2 + Y^2) - w^2\big) \tag{13}$$

satisfies $\nabla_{(X,Y,w)}\cdot\mathbf{F}_\omega = 0$, $\nabla B\cdot\mathbf{F}_\omega = 2w(1-B)$, and $\nabla\Psi\cdot\mathbf{F}_\omega = 0$, and leaves the midplane transfers (7) *unchanged. When $r > 0$ and $\omega \neq 0$, the projected logarithmic spiral slope is $\mathrm{d}\log r/\mathrm{d}\lambda = w/\omega$, reversing under $w \mapsto -w$. On each shell $0 < a < 1/2$, the rectified meridional angle $\psi(\theta) = \frac{2\pi}{P(a)}\int_0^\theta (2\sqrt{1/2 - a\cos s})^{-1}\,\mathrm{d}s \in \mathbb{R}/2\pi\mathbb{Z}$ conjugates the nonlinear flow of $\mathbf{F}_\omega$ through Theorem A0 to the linear Kronecker flow $\bar\Phi_\tau(\psi,\lambda) = (\psi + \frac{2\pi}{P(a)}\tau,\ \lambda + \omega\tau)$ on $\mathbb{T}^2$. Its equatorial Poincaré return map $\lambda \mapsto \lambda + \omega P(a) \pmod{2\pi}$ closes after $m \geq 1$ meridional periods if and only if $\frac{\omega P(a)}{2\pi} \in \mathbb{Q}$.*

Proof. The rotational perturbation $(\omega Y, -\omega X, 0)$ is divergence-free, has zero w-component (leaving $J_z(r, 0)$ unchanged), and is orthogonal to $(2X, 2Y, 0) = \nabla_{(X,Y,0)}u$, hence preserves u, $B = u + w^2$, and $\Psi(u, w)$. With $X = \sqrt{u}\sin\lambda$, $Y = \sqrt{u}\cos\lambda$, $\dot\lambda = \omega$ and $\dot r/r = \dot u/(2u) = w$, giving $\mathrm{d}\log r/\mathrm{d}\lambda = w/\omega$. Along $\dot\theta = 2\sqrt{1/2 - a\cos\theta}$, $\dot\psi = \frac{2\pi}{P(a)}(2\sqrt{u})^{-1}\dot\theta = \frac{2\pi}{P(a)}$, establishing the exact conjugation to $\bar\Phi_\tau$. After $m \geq 1$ meridional cycles ($\tau = mP(a)$), ψ returns to $\psi + 2\pi m \equiv \psi$, while λ returns modulo 2π if and only if $m\omega P(a) = 2\pi k$ for some $k \in \mathbb{Z}$, which is equivalent to $\frac{\omega P(a)}{2\pi} = k/m \in \mathbb{Q}$. □

3 Heat turns the water

Heat enters along the first arrow of (4), alongside a magnetic field acting in two ways. At uniform susceptibility its Kelvin force [52] is a gradient, setting scalar pressure without disturbing vorticity and supplying the junction bias for NOR or NAND (Theorem A6). When susceptibility varies with temperature, crossed gradients of temperature and magnetic energy give the Kelvin force a curl, driving thermomagnetic convection [14]; for axisymmetric profiles that curl aligns with the circulation's vorticity and turns the water (Theorem A7). Schelling saw nature moved by opposed forces: nature is visible spirit and spirit invisible nature [54]. Any two-bath cycle of positive affinity [56] delivers its work inside the Carnot bound.

Theorem A6 (The field sets the pressure)**.** *Let $\phi(\mathbf{x}) = |\mathbf{B}(\mathbf{x})|^2$ be a C^2 squared magnetic-flux density on Ω, with $\mu_0 \neq 0$ and spatially constant magnetic susceptibility $\chi_m \in \mathbb{R}$. The Kelvin body force $\mathbf{f}_K = \frac{\chi_m}{2\mu_0}\nabla\phi$ is curl-free, $\nabla \times \mathbf{f}_K = \mathbf{0}$. Acting on an isothermal homogeneous fluid, the static magnetic field preserves the velocity and bulk vorticity of the Navier–Stokes closure* (11) *while setting its scalar pressure to $p_{\text{eff}} = p_\mu - \frac{\chi_m}{2\mu_0}|\mathbf{B}|^2$.*

Proof. Because $\phi \in C^2(\Omega)$, mixed partial derivatives commute ($\partial_y\partial_z\phi = \partial_z\partial_y\phi$, etc.), and constancy of $c = \chi_m/(2\mu_0)$ gives $(\nabla \times (c\nabla\phi))_x = c(\partial_y\partial_z\phi - \partial_z\partial_y\phi) = 0$, and identically for y, z. Adding $\mathbf{f}_K = \nabla(\frac{\chi_m}{2\mu_0}\phi)$ to the right-hand side of (11) absorbs directly into $-\nabla p_{\text{eff}}$. □

Theorem A7 (Heat crossed with the field)**.** *Let $\chi_m(\mathbf{x}) = \hat{\chi}(T(\mathbf{x}))$ be a C^1 thermally (or compositionally) stratified susceptibility and $\phi(\mathbf{x}) = |\mathbf{B}(\mathbf{x})|^2$ a C^2 magnetic energy scalar.*

(a) The Kelvin force has curl

$$\nabla \times \left(\frac{\chi_m}{2\mu_0}\nabla\phi\right) = \frac{1}{2\mu_0}\big(\nabla\chi_m \times \nabla\phi\big) = \frac{\hat{\chi}'(T)}{2\mu_0}\big(\nabla T \times \nabla\phi\big). \tag{14}$$

For axisymmetric profiles $\chi_m = \tilde{\chi}(u, z)$ and $\phi = \tilde{\phi}(u, z)$ ($u = (x^2 + y^2)/R^2$, $R \neq 0$), (14) *points along the azimuth,*

$$\nabla \times \mathbf{f}_K = \frac{1}{\mu_0 R^2}\big(\partial_z\tilde{\chi}\,\partial_u\tilde{\phi} - \partial_u\tilde{\chi}\,\partial_z\tilde{\phi}\big)(-y,\ x,\ 0), \tag{15}$$

matching the Theorem A4 vorticity axis $(-y, x, 0)$ wherever axial thermal stratification $\partial_z\tilde{\chi} \neq 0$ crosses radial magnetic variation $\partial_u\tilde{\phi} \neq 0$, and vanishing when $\nabla\tilde{\chi}$ and $\nabla\tilde{\phi}$ are collinear.

(b) For any quantized or continuum two-bath thermodynamic cycle exchanging energy increments $\Delta_h > 0$ with a hot bath at $T_h > 0$ and Δ_c with a cold bath at $T_c > 0$ to deliver modal work $\Delta_w = \Delta_h - \Delta_c$, n-cycle conservation reads $n\Delta_h = n\Delta_c + n\Delta_w$ ($n \in \mathbb{Z}$). Positive cycle affinity $\frac{\Delta_c}{T_c} - \frac{\Delta_h}{T_h} > 0$ holds if and only if $\Delta_c T_h - \Delta_h T_c > 0$, which gives the strict Carnot bound

$$\frac{\Delta_w}{\Delta_h} < 1 - \frac{T_c}{T_h}. \tag{16}$$

Proof. (a) By the product rule and symmetry of second partials of ϕ,

$$\partial_y(k\chi_m\partial_z\phi)-\partial_z(k\chi_m\partial_y\phi) = k(\partial_y\chi_m\,\partial_z\phi-\partial_z\chi_m\,\partial_y\phi)+k\chi_m(\partial_y\partial_z\phi-\partial_z\partial_y\phi) = k(\nabla\chi_m\times\nabla\phi)_x$$

with $k=(2\mu_0)^{-1}$, and cyclic permutation gives (14). When $\chi_m=\tilde{\chi}(u,z)$ and $\phi=\tilde{\phi}(u,z)$, $\nabla\chi_m = (\frac{2x}{R^2}\partial_u\tilde{\chi},\ \frac{2y}{R^2}\partial_u\tilde{\chi},\ \partial_z\tilde{\chi})$ and similarly for $\nabla\phi$. Their cross product has z-component $\frac{4xy}{R^4}(\partial_u\tilde{\chi}\partial_u\tilde{\phi}-\partial_u\tilde{\chi}\partial_u\tilde{\phi})=0$, x-component $\frac{2y}{R^2}(\partial_u\tilde{\chi}\partial_z\tilde{\phi}-\partial_z\tilde{\chi}\partial_u\tilde{\phi}) = \frac{2}{R^2}(\partial_z\tilde{\chi}\partial_u\tilde{\phi}-\partial_u\tilde{\chi}\partial_z\tilde{\phi})(-y)$, and y-component $\frac{2}{R^2}(\partial_z\tilde{\chi}\partial_u\tilde{\phi}-\partial_u\tilde{\chi}\partial_z\tilde{\phi})x$, dividing by $2\mu_0$ to yield (15).

(b) Distributivity gives $n\Delta_h = n\Delta_c+n(\Delta_h-\Delta_c)$. Since $T_cT_h>0$, $\frac{\Delta_c}{T_c}-\frac{\Delta_h}{T_h}=\frac{\Delta_cT_h-\Delta_hT_c}{T_cT_h}>0 \iff \Delta_cT_h-\Delta_hT_c>0$. Dividing $\Delta_cT_h-\Delta_hT_c>0$ by $\Delta_hT_h>0$ gives $\frac{\Delta_c}{\Delta_h}>\frac{T_c}{T_h}$, so $\frac{\Delta_w}{\Delta_h}=1-\frac{\Delta_c}{\Delta_h}<1-\frac{T_c}{T_h}$. □

An azimuthal Lorentz force stays orthogonal to $\nabla\Psi$, driving swirl without crossing a shell. The swirl's centrifugal acceleration $-\frac{v_\varphi^2}{r}\hat{\mathbf{e}}_r$ absorbs into pressure and leaves meridional vorticity intact whenever the squared spin is constant in height; where squared spin varies with height, the centrifugal curl drives the secondary meridional flow of electromagnetic stirring [10]. Any smooth nonzero mirror-odd swirl vanishes at the midplane and therefore varies with height, feeding back into the meridional balance.

Theorem A8 (The swirl the circulation keeps)**.** *For an axisymmetric radial conduction current $\mathbf{J}_e(x,y,z)=j_e(u,z)(x,y,0)$ in a uniform axial magnetic field $\mathbf{B}=(0,0,B_0)$, let $\mathbf{f}_L=\mathbf{J}_e\times\mathbf{B}=B_0j_e(u,z)(y,-x,0)$ be the Lorentz body force.*

(a) For every differentiable radial profile $u\mapsto j_e(u,z)$, the force $\mathbf{f}_L$ is kinematically orthogonal to the shells and boundary: $\nabla\cdot\mathbf{f}_L=0$, $\nabla B\cdot\mathbf{f}_L=0$, and $\nabla\Psi\cdot\mathbf{f}_L=0$ throughout $\mathbb{R}^3$. Under $z\mapsto -z$, $\mathbf{f}_L$ is mirror-even (pure drive) when $j_e(u,-z)=j_e(u,z)$ and mirror-odd (pure torsion) when $j_e(u,-z)=-j_e(u,z)$.

(b) For any axisymmetric azimuthal velocity field $\mathbf{v}_\varphi(x,y,z)=\Omega_s(u,z)(-y,x,0)$ with C^1 squared spin $S(u,z)=\Omega_s(u,z)^2$, the radial centrifugal acceleration $\mathbf{a}_{\text{cent}}(x,y,z)=-S(u,z)(x,y,0)$ has curl

$$\nabla\times\mathbf{a}_{\text{cent}} = -\big(\partial_zS(u,z)\big)(-y,\ x,\ 0). \tag{17}$$

Off the symmetry axis ($(x,y)\neq(0,0)$), $\nabla\times\mathbf{a}_{\text{cent}}=\mathbf{0}$ if and only if $\partial_z(\Omega_s^2)=0$. In particular, a z-independent rigid swirl $\Omega_s=$ const (Theorem A5) yields the conservative centrifugal gradient $-\nabla(\frac{1}{2}\Omega_s^2r^2)$, which absorbs into scalar pressure while preserving meridional vorticity and Ψ.

(c) If, for some $u>0$, the map $z\mapsto\Omega_s(u,z)$ is mirror-odd ($\Omega_s(u,-z)=-\Omega_s(u,z)$), nonzero at some height $z_1>0$, and $z\mapsto\Omega_s(u,z)^2$ is differentiable on $[0,z_1]$, then there exists $\xi\in(0,z_1)$ with $\partial_z(\Omega_s^2)(u,\xi)\neq 0$; on the circle $x^2+y^2=R^2u$ at height ξ, $\nabla\times\mathbf{a}_{\text{cent}}\neq\mathbf{0}$, coupling azimuthal torsion back into the meridional momentum balance.

Proof. (a) Since $\mathbf{f}_L = B_0 j_e(u,z)(y,-x,0)$ has zero z-component and $(x,y)\cdot(y,-x) = 0$, $\nabla B \cdot \mathbf{f}_L = 0$ and $\nabla\Psi \cdot \mathbf{f}_L = 0$ hold pointwise. Differentiating gives

$$\partial_x(B_0 j_e(u,z)y) + \partial_y(-B_0 j_e(u,z)x) = B_0 \partial_u j_e \frac{2xy}{R^2} - B_0 \partial_u j_e \frac{2yx}{R^2} = 0.$$

Parity under $z \mapsto -z$ follows directly from $j_e(u,-z) = \pm j_e(u,z)$.

(b) Write $\mathbf{a}_{\text{cent}} = (-S(u,z)x,\, -S(u,z)y,\, 0)$. Its curl has z-component

$$\partial_x(-Sy) - \partial_y(-Sx) = -\frac{2xy}{R^2}\partial_u S + \frac{2yx}{R^2}\partial_u S = 0,$$

even when $\partial_u S \neq 0$. Its x- and y-components are $0 - \partial_z(-Sy) = (\partial_z S)y = -(\partial_z S)(-y)$ and $\partial_z(-Sx) - 0 = -(\partial_z S)x$, proving (17). When $(x,y) \neq (0,0)$, $(-y,x) \neq (0,0)$, so (17) vanishes iff $\partial_z S = 0$. When Ω_s is constant, $\mathbf{a}_{\text{cent}} = -\nabla(\frac{1}{2}\Omega_s^2(x^2+y^2))$.

(c) Mirror oddness at $z = 0$ forces $\Omega_s(u,0) = -\Omega_s(u,0)$, so $\Omega_s(u,0) = 0$, whereas $\Omega_s(u,z_1)^2 > 0$. Applying the Mean Value Theorem to $z \mapsto \Omega_s(u,z)^2$ on $[0,z_1]$ gives $\xi \in (0,z_1)$ with $\partial_z(\Omega_s^2)(u,\xi) = \Omega_s(u,z_1)^2/z_1 > 0$. By (b), $\nabla \times \mathbf{a}_{\text{cent}}(x,y,\xi) \neq \mathbf{0}$ whenever $x^2+y^2 = R^2 u$. □

The swirl of Theorem A5 is held by this torsion. With the rotation $(U\omega/H)(y,-x,0)$ added to the circulation, the steady momentum balance closes under the azimuthal body force $\frac{2\rho U^2 \omega z}{H^3}(y,-x,0)$, odd in z: the Lorentz force of the radial current $j_e = \frac{2\rho U^2 \omega z}{B_0 H^3}$ in the axial field of Theorem A8, a pure torsion.

4 Twelve vessels

Set $N \geq 1$ identical vessels on the cyclic group $\mathbb{Z}/N\mathbb{Z}$, a ring without corners [41] and the standard network of identical units with dihedral symmetry [15]. Shift-invariant coupling around the ring forms a circulant [11], and the time operator decides how each eigenvalue acts. On a symmetric restoring ring, a first-order operator relaxes each mode at a real positive rate without oscillation, whereas inertia and compliance bring the ring to resonance [37], turning each level into a frequency $\Omega_k = \sqrt{\Lambda_k/M}$. At $N = 12$, the wheel of the twelve signs, a nonzero symmetric coupling between neighbours folds the twelve modes onto seven levels.

Theorem A9 (Twelve vessels, seven levels)**.** *Let $N \geq 1$ and define the shift-invariant spatial operator $(K\mathbf{q})_j = \sum_{m=0}^{N-1} c_m q_{(j+m) \bmod N}$ on $\mathbb{C}^N$ ($c_0, \ldots, c_{N-1} \in \mathbb{C}$).*

(a) For every $\zeta \in \mathbb{C}$ with $\zeta^N = 1$ (in particular $\zeta_{N,k} = e^{2\pi i k/N}$, $0 \leq k < N$), the discrete Fourier mode $v_j^{(\zeta)} = \zeta^j$ is an eigenvector of K, $(Kv^{(\zeta)})_j = \Lambda(\zeta) v_j^{(\zeta)}$, with spatial eigenvalue $\Lambda(\zeta) = \sum_{m=0}^{N-1} c_m \zeta^m$.

(b) On $\mathbb{Z}/12\mathbb{Z}$, rotation $r(x) = x+1$ (order 12) and reflection $s(x) = -x$ (order 2) satisfy $srs = r^{-1}$ and generate the 24-element dihedral group D_{12}. For nearest-neighbor

symmetric couplings ($c_0 \in \mathbb{R}$, $c_1 = c_{11} \in \mathbb{R}$, others 0), $\Lambda_k = c_0 + 2c_1\cos(\pi k/6)$ takes seven real levels $\{c_0 \pm 2c_1,\ c_0 \pm \sqrt{3}c_1,\ c_0 \pm c_1,\ c_0\}$. In particular, on a restoring lattice $K = k_0 I - \kappa\Delta_{\mathbb{Z}/12}$ with local stiffness $k_0 > 0$ and coupling $\kappa \geq 0$ ($c_0 = k_0 + 2\kappa$, $c_1 = c_{11} = -\kappa$), every spatial eigenvalue is positive:

$$\Lambda_k = k_0 + 2\kappa\left(1 - \cos\frac{\pi k}{6}\right) = k_0 + 4\kappa\sin^2\frac{\pi k}{12} \geq k_0 > 0 \qquad (0 \leq k < 12). \tag{18}$$

(c) First-order relaxation contrasts with second-order modal resonance:

- *Under first-order dissipative evolution $C_h\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{0}$ ($C_h > 0$, $\Lambda_k \in \mathbb{R}$), each Fourier amplitude obeys $\hat{q}_k(t) = e^{-(\Lambda_k/C_h)t}\hat{q}_k(0)$, yielding real exponential decay or growth without oscillation ($\hat{q}_k(t) = 0 \iff \hat{q}_k(0) = 0$).*
- *Under second-order inertial-compliant evolution $M\ddot{\mathbf{q}} + \Gamma\dot{\mathbf{q}} + K\mathbf{q} = \mathbf{s}(t)$ ($M > 0$, $\Lambda_k > 0$, $\Gamma \geq 0$), each mode obeys $M\ddot{\hat{q}}_k + \Gamma\dot{\hat{q}}_k + \Lambda_k\hat{q}_k = \hat{s}_k(t)$ with undamped eigenfrequency $\Omega_k = \sqrt{\Lambda_k/M} > 0$. At $\Gamma = 0$, the phase-space map*

$$\Phi_t^{(k)}(\hat{q}, \hat{p}) = \left(\hat{q}\cos(\Omega_k t) + \frac{\hat{p}}{\Omega_k}\sin(\Omega_k t),\ -\Omega_k\hat{q}\sin(\Omega_k t) + \hat{p}\cos(\Omega_k t)\right) \tag{19}$$

is a flow semigroup solving $\dot{\hat{q}} = \hat{p}$, $M\dot{\hat{p}} + \Lambda_k\hat{q} = 0$ and conserving modal energy $E_k(\hat{q}, \hat{p}) = \frac{1}{2}M\hat{p}^2 + \frac{1}{2}\Lambda_k\hat{q}^2$. When $\Gamma > 0$, harmonic forcing $e^{i\omega t}$ ($\omega \geq 0$) encounters squared impedance $|Z_k(\omega)|^2 = (\Lambda_k - M\omega^2)^2 + (\Gamma\omega)^2 > 0$ everywhere on $\mathbb{R}$, whose reactive part $\Lambda_k - M\omega^2$ vanishes on $\omega \geq 0$ if and only if $\omega = \Omega_k = \sqrt{\Lambda_k/M}$.

Proof. (a) Write $j + m = qN + ((j+m) \bmod N)$; since $\zeta^N = 1$, $\zeta^{(j+m) \bmod N} = \zeta^{j+m} = \zeta^m\zeta^j$, so $(Kv^{(\zeta)})_j = \sum_m c_m\zeta^m\zeta^j = \Lambda(\zeta)v_j^{(\zeta)}$.

(b) The D_{12} relations hold in $\mathbb{Z}/12\mathbb{Z}$ since $12 \cdot 1 \equiv 0$, $-(-x) = x$ with $-1 \equiv 11 \not\equiv 1$, and $-(-x+1) = x-1$ [41, Theorem W8]. At $N = 12$, $\zeta_{12,k} + \zeta_{12,k}^{11} = e^{i\pi k/6} + e^{-i\pi k/6} = 2\cos(\pi k/6)$. Evaluating $\cos(\pi k/6)$ for $k = 0, \ldots, 11$ gives $\{1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1\}$ and its reflection, yielding the seven exact levels. Since $\cos(\pi k/6) \leq 1$ and $1 - \cos(2\alpha) = 2\sin^2\alpha$, (18) holds with $\Lambda_k \geq k_0 > 0$.

(c) Differentiating $\hat{q}_k(t) = e^{-(\Lambda_k/C_h)t}\hat{q}_k(0)$ gives $C_h\dot{\hat{q}}_k = -\Lambda_k\hat{q}_k$, and $e^{-(\Lambda_k/C_h)t} > 0$ precludes sign changes or zero crossings when $\hat{q}_k(0) \neq 0$. For the second-order operator at $\Gamma = 0$, $\Omega_k = \sqrt{\Lambda_k/M} > 0$ satisfies $\Omega_k^2 = \Lambda_k/M$. Angle-addition identities for $\cos(\Omega_k(t+s))$ and $\sin(\Omega_k(t+s))$ verify $\Phi_{t+s}^{(k)} = \Phi_t^{(k)} \circ \Phi_s^{(k)}$ and $\Phi_0^{(k)} = \mathrm{id}$; differentiation gives $\dot{\hat{q}} = \hat{p}$ and $\dot{\hat{p}} = -\Omega_k^2\hat{q} = -(\Lambda_k/M)\hat{q}$, while substituting $M\Omega_k^2 = \Lambda_k$ and $\sin^2 + \cos^2 = 1$ yields $E_k(\Phi_t^{(k)}(\hat{q}, \hat{p})) = E_k(\hat{q}, \hat{p})$. Finally, when $\Gamma > 0$, $|Z_k(0)|^2 = \Lambda_k^2 > 0$ and $|Z_k(\omega)|^2 \geq (\Gamma\omega)^2 > 0$ for $\omega \neq 0$; for $\omega \geq 0$, $\Lambda_k - M\omega^2 = 0 \iff \omega^2 = \Lambda_k/M \iff \omega = \sqrt{\Lambda_k/M} = \Omega_k$. □

5 The trit where three waters meet

Theorem A0(d) splits the ring of twelve through μ_{12} into $\mathsf{Trit} \times \mathrm{Fin}\,4$, mode and element, where $\mathsf{Trit} = \mathrm{Fin}\,3 = \{0, 1, 2\}$. The same three states form the trit kept by every ordered

decision [40]: kill, propagate and generate [60]. Proposition 5.1 records their algebra.

Proposition 5.1 (The algebra of the trit)**.** *On* $\mathsf{Trit} = \{0, 1, 2\}$*:*

(a) Cyclic shift $\mathrm{shift}(x, k) = (x + k) \bmod 3$ *is associative of period* 3*, generated by two involution swaps* $\sigma_{01} \circ \sigma_{12}(x) = (x + 1) \bmod 3$*, and represented by orthogonal* 3×3 *integer permutation matrices* $P_k^\top P_k = I_3$*,* $P_k P_\ell = P_{k+\ell}$*.*

(b) The carry unifier $\mathrm{unify}(x, y) = y$ *if* $x = 1$ *else* x *[40, Theorem T9] is associative with two-sided identity* 1*, is non-injective (*$\mathrm{unify}(0, 0) = \mathrm{unify}(0, 2)$*), composes homomorphically on Markov column channels* $M_{\mathrm{unify}(x)} M_{\mathrm{unify}(y)} = M_{\mathrm{unify}(\mathrm{unify}(x,y))}$*, and lifts to a reversible bijection on* Trit^3*:*

$$\mathcal{L}(x, y, z) = \big(x,\ y,\ (z + \mathrm{unify}(x, y)) \bmod 3\big),$$
$$\mathcal{L}^{-1}(x, y, z) = \big(x,\ y,\ (z - \mathrm{unify}(x, y)) \bmod 3\big).$$

(c) In the 3×3 *skew-Hermitian Lie algebra* $\mathfrak{su}(3)$ *with real antisymmetric generators* $A_{ij} = E_{ij} - E_{ji}$*, imaginary symmetric generators* $iB_{ij} = i(E_{ij} + E_{ji})$*, and split drift* $D_{\mathrm{drift}} = i(-2E_{00} + E_{11} + E_{22})$*, the commutators satisfy* $[iB_{01}, iB_{12}] = -A_{02}$*,* $[D_{\mathrm{drift}}, iB_{01}] = 3A_{01}$*,* $[D_{\mathrm{drift}}, iB_{12}] = 0$*, and* $[A_{01}, iB_{12}] = iB_{02}$*, while signed rank-one outer products* $v_{ij}^+ (v_{ij}^+)^\top - v_{ij}^- (v_{ij}^-)^\top = 2(E_{ij} + E_{ji})$ *(*$v_{ij}^\pm = e_i \pm e_j$*,* $i \neq j$*) supply traceless pure-shear operators.*

(d) No injective map $b : \mathsf{Trit} \to \{0, 1\}$ *exists; no one-bit readout carries the native three-state algebra without loss.*

Proof. Parts (a)–(c) are finite algebraic identities on $\mathrm{Fin}\, 3$ and $\mathbb{Z}^{3\times 3}$ verified exhaustively by direct expansion (`TernaryAlgebra.lean`). Part (d) follows immediately from $|\mathsf{Trit}| = 3 > 2 = |\{0, 1\}|$ by the pigeonhole principle (Theorem A0(c)). □

By part (d) the trit needs three levels, and a junction pressure supplies them: where three branches meet, the node settles onto one of three distinct pressures, and the scalar bias of Theorem A6 sets the threshold that reads those levels as a gate. A meeting of three ways is the *triodos*, and in Greek cult the herms of Hermes, god of roads, boundaries and thresholds, stood at crossroads [7].

Theorem A10 (The junction keeps the trit)**.** *Join two binary pressure inputs* $P_0 + a\Delta P$*,* $P_0 + b\Delta P$ *(*$a, b \in \{0, 1\}$*,* $\Delta P > 0$*) and one reference reservoir at pressure* P_0 *through three equal nonzero linear conductances* $G \neq 0$*, together with a static diamagnetic Kelvin interface bias* $\Delta P_B = -\frac{\chi_m}{2\mu_0}(B_{\mathrm{jun}}^2 - B_{\mathrm{ref}}^2)$ *from Theorem A6.*

(a) Unloaded conservation $G(P_0 + a\Delta P - P_j) + G(P_0 + b\Delta P - P_j) + G(P_0 - P_j) = 0$ *holds if and only if*

$$P_j - P_0 = \frac{a + b}{3}\Delta P \in \left\{0,\ \frac{1}{3}\Delta P,\ \frac{2}{3}\Delta P\right\}, \tag{20}$$

which stands in order-preserving bijection with the column-sum carry trit $t(a,b) = a+b = (a \oplus b) + 2ab \in \mathsf{Trit}$ *(kill* $= 0$*, propagate* $= 1$*, generate* $= 2$*).*

(b) A strict threshold detector $D_{<P_c}(P_j) = [(P_j - P_0) + \Delta P_B < P_c]$ *computes* $\mathrm{NAND}(a,b) = 1 - ab$ *for all* $(a,b) \in \{0,1\}^2$ *if and only if*

$$\frac{1}{3}\Delta P < P_c - \Delta P_B \leq \frac{2}{3}\Delta P. \tag{21}$$

At the upper endpoint $P_c - \Delta P_B = \frac{2}{3}\Delta P$ *the detector realizes* NAND*, whereas at the lower endpoint* $P_c - \Delta P_B = \frac{1}{3}\Delta P$ *it fails on* $(0,1)$ *and* $(1,0)$*. Requiring a bilateral noise margin* $\delta > 0$ *(*$P_j - P_0 + \Delta P_B \leq P_c - \delta$ *when* $ab = 0$*,* $\geq P_c + \delta$ *when* $ab = 1$*) tightens* (21) *to* $\frac{1}{3}\Delta P + \delta \leq P_c - \Delta P_B \leq \frac{2}{3}\Delta P - \delta$*, reaching its maximum margin* $\delta_{\max} = \frac{1}{6}\Delta P$ *uniquely at* $P_c - \Delta P_B = \frac{1}{2}\Delta P$*.*

(c) Shifting the scalar Kelvin bias ΔP_B *into the adjacent half-open* NOR *window*

$$0 < P_c - \Delta P_B \leq \frac{1}{3}\Delta P \tag{22}$$

reconfigures the same junction to compute $\mathrm{NOR}(a,b) = (1-a)(1-b)$ *on all* $(a,b) \in \{0,1\}^2$*, and conversely every such* NOR *readout lies in* (22)*.*

(d) Composing four ideally restored NAND *projections via* $c = \mathrm{NAND}(a,b)$*,* $d = \mathrm{NAND}(a,c)$*,* $e = \mathrm{NAND}(b,c)$*,* $y = \mathrm{NAND}(d,e)$ *evaluates the parity* $y = a \oplus b$*, while* $(t(a,b) - y)/2 = ab$ *recovers the carry bit.*

Proof. (a) Factoring the Kirchhoff sum gives $G\big((a+b)\Delta P - 3(P_j - P_0)\big) = 0$; since $G \neq 0$, (20) holds uniquely. Because $\Delta P/3 > 0$, $(a_1 + b_1) \leq (a_2 + b_2) \iff \frac{a_1+b_1}{3}\Delta P \leq \frac{a_2+b_2}{3}\Delta P$, and $a + b = (a \oplus b) + 2ab$ holds across $\{0,1\}^2$ [40, Theorem T5].

(b) Put $\theta = P_c - \Delta P_B$. Then $D_{<P_c}(P_j(a,b)) = [\frac{a+b}{3}\Delta P < \theta]$. To equal $1 - ab$ on all four pairs, $(0,0)$ requires $0 < \theta$, $(0,1)$ and $(1,0)$ require $\frac{1}{3}\Delta P < \theta$ (which implies $0 < \theta$ since $\Delta P > 0$), and $(1,1)$ requires $\neg(\frac{2}{3}\Delta P < \theta)$, which is $\theta \leq \frac{2}{3}\Delta P$. Thus (21) is both necessary and sufficient. At $\theta = \frac{2}{3}\Delta P$, $\frac{2}{3}\Delta P < \frac{2}{3}\Delta P$ is false (output 0) while $\frac{1}{3}\Delta P < \frac{2}{3}\Delta P$ is true (output 1); at $\theta = \frac{1}{3}\Delta P$, $\frac{1}{3}\Delta P < \frac{1}{3}\Delta P$ is false on $(0,1)$. The margin inequalities $\max(0, \frac{1}{3}\Delta P) \leq \theta - \delta$ and $\frac{2}{3}\Delta P \geq \theta + \delta$ rearrange directly to $\frac{1}{3}\Delta P + \delta \leq \theta \leq \frac{2}{3}\Delta P - \delta$.

(c) Similarly, $\mathrm{NOR}(a,b) = 1$ at $(0,0)$ iff $0 < \theta$, and $\mathrm{NOR}(a,b) = 0$ on $(0,1), (1,0), (1,1)$ iff $\neg(\frac{1}{3}\Delta P < \theta)$ and $\neg(\frac{2}{3}\Delta P < \theta)$, which reduces to $\theta \leq \frac{1}{3}\Delta P$ since $\Delta P > 0$.

(d) Case check on $\{0,1\}^2$: for $(0,0)$, $c = 1, d = 1, e = 1, y = 0$; for $(0,1)$, $c = 1, d = 1, e = 0, y = 1$; for $(1,0)$, $c = 1, d = 0, e = 1, y = 1$; for $(1,1)$, $c = 0, d = 1, e = 1, y = 0$. Thus $y = a \oplus b$, and $(a + b - (a \oplus b))/2 = 2ab/2 = ab$. □

6 One factorization

Every arrow of (4) composes into a single structure, gathered in Theorem A11: one vessel contracts to its torus, twelve vessels on a ring contract to their spectrum and their trit, and

every threshold gate of the junction factors through a reading of that trit.

Theorem A11 (Vessel, ring and gate)**.** *Let Ω be the ellipsoid* (5) *$(R, H, \rho > 0)$ replicated around the $N = 12$ dihedral ring with inertance $M > 0$, damping $\Gamma > 0$, local stiffness $k_0 > 0$, and coupling $\kappa \geq 0$.*

(a) On every shell $0 < a < 1/2$ with elliptic period $P(a) > 0$ and constant swirl $\omega \in \mathbb{R}$ (decoupled from meridional vorticity by Theorem A8(b)), the Kronecker flow $\bar{\Phi}_\tau(\psi, \lambda) = (\psi + \frac{2\pi}{P(a)}\tau, \lambda + \omega\tau)$ is a flow semigroup whose n-period stroboscopic azimuthal coordinate contracts equivariantly to $\lambda \mapsto \lambda + n\omega P(a)$ and closes in finite time if and only if $\frac{\omega P(a)}{2\pi} \in \mathbb{Q}$.

(b) Every ring mode $k \in \operatorname{Fin} 12$ carries a positive restoring eigenvalue $\Lambda_k \geq k_0 > 0$ and eigenfrequency $\Omega_k = \sqrt{\Lambda_k / M} > 0$, generating an energy-conserving Hamiltonian semigroup $\Phi_t^{(k)}$ at $\Gamma = 0$ and a nonsingular resonant impedance $|Z_k(\omega)|^2 > 0$ for $\Gamma > 0$, while $\mu_{12} : \operatorname{Fin} 12 \xrightarrow{\sim} \mathsf{Trit} \times \operatorname{Fin} 4$ contracts the twelve-sector ring canonically onto a ternary $\times$ quaternary product state.

(c) With binary pressure encoding $E_{\Delta P}(a, b) = (P_0 + a\Delta P, P_0 + b\Delta P)$ $(\Delta P > 0)$, magnetically biased junction evolution $\Phi_{\Delta P_B}(p_1, p_2) = \frac{1}{3}((p_1 - P_0) + (p_2 - P_0)) + \Delta P_B$, and threshold readout $D_{P_c}(p_{\text{eff}}) = [p_{\text{eff}} < P_c]$, every scalar bias $\Delta P_B \in \mathbb{R}$ gives the factorization

$$D_{P_c} \circ \Phi_{\Delta P_B} \circ E_{\Delta P} = \bar{D}_{\Delta P, \Delta P_B, P_c} \circ t_{\text{carry}}, \tag{23}$$

where $t_{\text{carry}} : \{0,1\}^2 \twoheadrightarrow \mathsf{Trit}$, $(a, b) \mapsto a + b$, maps surjectively onto Trit (which admits no injective map to $\{0,1\}$), and the reduced readout $\bar{D}_{\Delta P, \Delta P_B, P_c}(t) = [\frac{t}{3}\Delta P + \Delta P_B < P_c]$ switches from NOR *on $0 < P_c - \Delta P_B \leq \frac{1}{3}\Delta P$ to* NAND *on $\frac{1}{3}\Delta P < P_c - \Delta P_B \leq \frac{2}{3}\Delta P$ by tuning the scalar Kelvin bias ΔP_B alone.*

Proof. Part (a) combines Theorem A0(a,b) with Theorem A5 and induction on n. Part (b) combines Theorem A9(b,c) with Theorem A0(d). For part (c), direct evaluation gives

$$(\Phi_{\Delta P_B} \circ E_{\Delta P})(a, b) = \frac{a\Delta P + b\Delta P}{3} + \Delta P_B = \frac{t_{\text{carry}}(a, b)}{3}\Delta P + \Delta P_B,$$

proving (23). Surjectivity of t_{carry} holds by $t_{\text{carry}}(0,0) = 0$, $t_{\text{carry}}(0,1) = 1$, $t_{\text{carry}}(1,1) = 2$; non-existence of a one-bit injection holds by Proposition 5.1(d); and the two exact bias equivalences follow from Theorem A10(b,c). □

7 The turn reads and writes

Track a reciprocal pair of parcels once around an interior stream surface and back to the seam; at the half-turn the two exchange places. As the surface narrows and widens, their

relative inertia $j(s)$ varies along the arc alongside a geometric coupling $g(s)$, and the second half of the turn repeats the first with g reversed. Every interior stream surface ($0 < a < \frac{1}{2}$) at every aspect ratio belongs to this reciprocal class (Theorem A12(b)).

Couple that turn weakly to a reservoir of oscillating modes, as in [61], and send the pair once around. Two observables return to the seam: the angle by which the turn has advanced, and the terminal amplitudes left in the modes. One positive geometric constant K stands in front of both. The return angle reads a relation among three phases that no pair of modes carries, the same triple-phase relation that the bispectrum detects in a wave field where the power spectrum is blind [16, 24]. Starting from independent phases, a small torsion writes a relation among three modes into the shifted amplitudes, with a prefactor opposite in sign to that of the read, while in the unshifted canonical coordinates that leading connected response cancels. At constant inertia, $K = 0$ and both leading channels fall silent. The variation of the inertia is the check that turns the motion into a return that can be read, Fichte's *Anstoß* meeting the outgoing activity of the I and reflecting it back upon itself [13], and K is its measure.

Definition 7.1 (Reciprocal symmetry, monotone metric class, and shape kernels)**.** Let $H > 0$ be a half-turn arc length and put $S = 2H$. Continuous profiles $j, g : \mathbb{R} \to \mathbb{R}$ have *reciprocal symmetry* on $[0, S]$ when

$$j(s + H) = j(s), \qquad j(H - s) = j(s), \qquad g(s + H) = -g(s), \qquad g(H - s) = -g(s) \tag{24}$$

for all $s \in \mathbb{R}$. They belong to the *monotone reciprocal metric class* $\mathcal{M}_{\mathrm{rec}}(H)$ when, in addition, $j(s) > 0$ for all $s \in \mathbb{R}$, j is strictly decreasing on $[0, H/2]$, and g is strictly increasing on $[0, H/2]$ with $g(0) < 0$ (whence $g(H/2) = 0$ and $-g(s) > 0$ on $(0, H/2)$). Define the primitive-variation kernel

$$\chi(s) = -s^2 j(s) + 2 \int_0^s t\, j(t)\, \mathrm{d}t, \tag{25}$$

the lag kernel $L_G(s) = s\, j(s) G(s) - \int_0^s j(t) G(t)\, \mathrm{d}t$ for any continuous profile $G : \mathbb{R} \to \mathbb{R}$, and the three shape scalars

$$K = \int_0^S j(s) g(s) \left(s\, \chi(s) + \int_0^s \chi(t)\, \mathrm{d}t \right) \mathrm{d}s, \tag{26}$$

$$Q_G = \int_0^S j(s) \left(\int_0^s L_G(t)\, \mathrm{d}t + s\, L_G(s) - \frac{1}{2} G(s) \chi(s) \right) \mathrm{d}s, \tag{27}$$

$$D = -\int_0^S \chi(s)\, \mathrm{d}s. \tag{28}$$

For launch momentum parameter $\mu \in \mathbb{R}$, seam force $F \in \mathbb{R}$, and coupling scale $\alpha \in \mathbb{R}$, define the quadratic internal-energy excess profile and leading non-affine return angle

$$P_\mu(s) = \frac{1}{2} j(s) (\mu + F s)^2 - F \int_0^s j(t)(\mu + F t)\, \mathrm{d}t, \tag{29}$$

$$\Delta\theta_{\mathrm{na}}(\alpha, \mu, F) = \alpha^6 \int_0^S j(s) g(s) \left((\mu + F s) P_\mu(s) + F \int_0^s P_\mu(t)\, \mathrm{d}t \right) \mathrm{d}s. \tag{30}$$

Definition 7.2 (Paired circulation chart and phase-torus observables)**.** For $0 < a < 1/2$, put $c = \frac{1}{4} - a^2 > 0$ and $D_a(\vartheta) = \sqrt{\frac{1}{4} - a^2\cos^2\vartheta} > 0$ ($\vartheta \in \mathbb{R}$), and define the paired stream-surface coordinates

$$u_a(\vartheta) = \frac{\sqrt{c}\,(1 - 2a\cos\vartheta)}{2D_a(\vartheta)}, \qquad w_a(\vartheta) = \frac{a\sin\vartheta}{\sqrt{D_a(\vartheta)\big(D_a(\vartheta) + \sqrt{c}\,\big)}}. \tag{31}$$

For parcel mass $m_p > 0$, equatorial radius $R > 0$, and semi-height $H_{\mathrm{cav}} > 0$, define the paired parcel metric and meridional arc length

$$\begin{aligned} M_{\mathrm{rec}}(\vartheta) &= m_p\left(R^2\left(u_a(\vartheta) + \frac{c}{u_a(\vartheta)}\right)\frac{u_a'(\vartheta)^2}{4u_a(\vartheta)^2} + 2H_{\mathrm{cav}}^2 w_a'(\vartheta)^2\right), \\ s_{\mathrm{rec}}(\vartheta) &= \int_0^{\vartheta}\sqrt{M_{\mathrm{rec}}(t)}\,\mathrm{d}t. \end{aligned} \tag{32}$$

For $r \in \mathbb{N}$ canonical oscillator pairs (Y_m, Z_m) of frequencies $\omega_m > 0$ and couplings $d_m = b_m\omega_m$, relative displacement $q \in \mathbb{R}$, canonical momentum $p_c \in \mathbb{R}$, mechanical momentum $p = p_c + \ell g(s)$, axial inertia $J_{\mathrm{ax}}(s) \neq 0$, and conserved axial torsion $\ell \in \mathbb{R}$, define the shifted energy coordinates $A_m = \omega_m(Y_m - d_m q/\omega_m^2) + iZ_m$, the unshifted canonical coordinates $B_m = \omega_m Y_m + iZ_m = A_m + b_m q$, and the internal arc energy

$$E_{\mathrm{int}}(s, q, p, A) = \frac{1}{2}j(s)p^2 + V(q) + \frac{1}{2}\sum_{m=0}^{r-1}|A_m|^2 + \frac{\ell^2}{2J_{\mathrm{ax}}(s)}. \tag{33}$$

For three modes ($r = 3$) with coupling weights $b \in \mathbb{R}^3$, launch radii $\rho \in \mathbb{R}^3$, phases $\varphi = (\varphi_0, \varphi_1, \varphi_2) \in \mathbb{R}^3$, and triad sum $\psi \in \mathbb{R}$, set $F_{b,\rho}(\varphi) = \sum_{m=0}^{2} b_m\rho_m\cos\varphi_m$, $\Phi_\psi(\varphi_1, \varphi_2) = (\varphi_1, \varphi_2, \varphi_1 + \varphi_2 + \psi)$, and initial amplitudes $A_m^{(0)}(\varphi) = \rho_m e^{-i\varphi_m}$, with torus averages $\langle f\rangle_{\mathbb{T}^2} = (2\pi)^{-2}\int_0^{2\pi}\int_0^{2\pi} f(\varphi_1, \varphi_2)\,\mathrm{d}\varphi_2\,\mathrm{d}\varphi_1$ and $\langle f\rangle_{\mathbb{T}^3} = (2\pi)^{-3}\int_{[0,2\pi]^3} f(\varphi)\,\mathrm{d}\varphi$. With linear phase-jitter matrices $u, v \in \mathbb{R}^{3\times 3}$ and deterministic shifts $\sigma \in \mathbb{C}^3$, define the leading shifted and unshifted torsion tangents at $\ell = 0$,

$$\delta A_m^{\mathrm{sh}}(\varphi) = -\alpha^5 b_m F_{b,\rho}(\varphi)^2 Q_g + iA_m^{(0)}(\varphi)\sum_{k=0}^{2}\big(u_{mk}\cos\varphi_k + v_{mk}\sin\varphi_k\big) + \sigma_m, \tag{34}$$

$$\delta A_m^{\mathrm{un}}(\varphi) = \delta A_m^{\mathrm{sh}}(\varphi) + \alpha^5 b_m F_{b,\rho}(\varphi)^2 Q_g, \tag{35}$$

and, using the centered tangent $\tilde{\delta}A_m(\varphi) = \delta A_m(\varphi) - \langle\delta A_m\rangle_{\mathbb{T}^3}$, the connected triple derivative

$$\mathcal{C}_3(A, \delta A) = \Big\langle\tilde{\delta}A_0\,A_1\overline{A_2} + A_0\,\tilde{\delta}A_1\overline{A_2} + A_0A_1\overline{\tilde{\delta}A_2}\Big\rangle_{\mathbb{T}^3}. \tag{36}$$

Theorem A12 (One constant, two channels)**.** *Let $H > 0$ and $S = 2H$.*

(a) Let j, g, J_{ax}, V be differentiable with $J_{\mathrm{ax}}(s) \neq 0$, and let $q(s), p(s), A_m(s)$ satisfy the arc-parametrized coupled equations at speed $v \neq 0$:

$$\begin{aligned} q'(s) &= \frac{j(s)p(s)}{v}, \qquad A_m'(s) = -\frac{i\omega_m}{v}A_m(s) - \frac{b_m j(s)p(s)}{v}, \\ p'(s) &= \ell g'(s) + \frac{-V'(q(s)) + \sum_{m=0}^{r-1} b_m\,\mathrm{Re}\,A_m(s)}{v}. \end{aligned} \tag{37}$$

The coupling power cancels out of the internal energy rate:

$$\frac{\mathrm{d}}{\mathrm{d}s}E_{\mathrm{int}}(s,q(s),p(s),A(s)) = \frac{1}{2}j'(s)p(s)^2 + \ell\, j(s)p(s)g'(s) - \frac{\ell^2 J'_{\mathrm{ax}}(s)}{2J_{\mathrm{ax}}(s)^2}. \tag{38}$$

For the autonomous Hamiltonian $H_{\mathrm{tot}} = \frac{1}{2}P_s^2 + E_{\mathrm{int}}(s,q,p_c+\ell g(s),A)$ *with real-analytic* $j,g,J_{\mathrm{ax}} > 0$*, even real-analytic* V *(*$V(0) = V'(0) = 0$*), fixed* $\omega_m > 0$*,* $b_m \in \mathbb{R}$*, and launch amplitudes* $A_m(0) = A_m^{(0)}(\varphi)$ *(*$r = 3$*,* $H_0 = \frac{1}{2}\sum_{m=0}^{2}\rho_m^2$*), set* $E_0 = E - H_0 > 0$ *and* $\alpha = (2E_0)^{-1/2}$*. Under the full-turn Poincaré return map* $W_{E,\ell}$ *at* $s = S$*, let* $\Delta\theta(S)$ *be the lifted azimuth increment at* $\ell = 0$ *from* $q(0) = 0$*,* $p_c(0) = \alpha\mu$*, compared against the affine reference return angle* $\theta_{\mathrm{aff}}(\alpha,\mu,\varphi)$ *with inverse speed* α *and linear restoring force* $V''(0)q$*. For canonical launch* $\underline{q(0) = p_c(0) = 0}$ *with independent uniform phases on* $\mathbb{T}^3$*, let* $\kappa_3(\ell) = \langle \tilde{A}_0(S,\ell)\tilde{A}_1(S,\ell)\tilde{A}_2(S,\ell)\rangle_{\mathbb{T}^3}$ *(*$\tilde{A}_m = A_m - \langle A_m\rangle_{\mathbb{T}^3}$*) be the connected third cumulant of the terminal shifted amplitudes. Reflection symmetry forces* $\kappa_3(0) = 0$*, and uniformly on compact* (μ,φ) *sets as* $\alpha \to 0^+$*,*

$$\begin{gathered}\Delta\theta(S) - \theta_{\mathrm{aff}}(\alpha,\mu,\varphi) = \Delta\theta_{\mathrm{na}}(\alpha,\mu,F_{b,\rho}(\varphi)) + O(\alpha^7),\\ \partial_\ell\big(e^{i\omega_m\alpha S}A_m(S,\ell)\big)\big|_{\ell=0} = \delta A_m^{\mathrm{sh}}(\varphi) + O(\alpha^6), \qquad \partial_\ell\kappa_3(0) = \mathcal{C}_3\big(A^{(0)},\delta A^{\mathrm{sh}}\big) + O(\alpha^6),\end{gathered} \tag{39}$$

while in unshifted canonical coordinates $B_m = A_m + b_m q$*,* $\partial_\ell(e^{i\omega_m\alpha S}\tilde{B}_m(S,\ell))\big|_{\ell=0} = \tilde{\delta}A_m^{\mathrm{un}}(\varphi) + O(\alpha^6)$ *and* $\partial_\ell\kappa_3^{\mathrm{un}}(0) = O(\alpha^6)$*. Under* $\hat{t} = t/\alpha$*,* $\hat{P}_s = \alpha P_s$*,* $\hat{p} = \alpha p$*, and* $\hat{\ell} = \alpha\ell$ *at fixed* s*, high excess energy* $E_0 = (2\alpha^2)^{-1}$ *is equivalent to unit reference speed* $\hat{v} = \alpha v = \sqrt{1 - 2\alpha^2(E_{\mathrm{int}} - H_0)}$ *with scaled frequencies* $\alpha\omega_m$ *and scaled seam coupling* $\alpha^2(-V'(q) + \sum_m b_m \operatorname{Re} A_m)$*. Moving the Poincaré section* $s = 0$ *to* $s = \sigma(x)$ *shifts any additive one-turn cocycle* h *by the coboundary* $a \circ W_{E,\ell} - a$ *[38] and leaves invariant cycle averages unchanged.*

(b) For all $m_p, R, H_{\mathrm{cav}} > 0$ *and* $0 < a < 1/2$ *(*$c = \frac{1}{4} - a^2$*), the paired curve* $(u_a(\vartheta), w_a(\vartheta))$ *stays globally on the interior stream surface* $\Psi(u_a(\vartheta), w_a(\vartheta)) = c$ *of Theorem A1 and executes the constituent-exchange half-turn*

$$u_a(\vartheta+\pi) = \frac{c}{u_a(\vartheta)}, \qquad w_a(\vartheta+\pi) = -w_a(\vartheta) \qquad (\vartheta \in \mathbb{R}). \tag{40}$$

The paired metric satisfies $M_{\mathrm{rec}}(\vartheta) > 0$ *for every* $\vartheta \in \mathbb{R}$*,* $s_{\mathrm{rec}} : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing global bijection with half-turn arc length* $H = s_{\mathrm{rec}}(\pi) > 0$ *and inverse* $\vartheta = s_{\mathrm{rec}}^{-1}$*, and the induced profiles*

$$j(s) = \frac{u_a(\vartheta(s)) + c/u_a(\vartheta(s))}{m_p R^2 c}, \qquad g(s) = \frac{u_a(\vartheta(s)) - c/u_a(\vartheta(s))}{2\big(u_a(\vartheta(s)) + c/u_a(\vartheta(s))\big)} = -a\cos\vartheta(s) \tag{41}$$

belong to $\mathcal{M}_{\mathrm{rec}}(H)$*. Under* $(m_p, R, H_{\mathrm{cav}}) \mapsto (\beta_m m_p, \beta_R R, \beta_R H_{\mathrm{cav}})$ *(*$\beta_m, \beta_R > 0$*),* K *is invariant, depending only on the aspect ratio* H_{cav}/R *and shell parameter* a*.*

(c) *For every pair (j,g) with the reciprocal symmetry* (24)*, one common scalar governs both channels through the cubic normal form*

$$Q_g = -\frac{K}{S}, \qquad \Delta\theta_{\mathrm{na}}(\alpha,\mu,F) = -\alpha^6\,\frac{K}{S}\,F^2\Big(\mu+\frac{SF}{2}\Big), \\ \frac{\partial}{\partial\mu}\Delta\theta_{\mathrm{na}}(\alpha,\mu,F)\Big|_{\mu=0} = -\alpha^6\,\frac{K}{S}\,F^2. \tag{42}$$

Whenever $j \in C^1(\mathbb{R})$, $\chi(s) = \int_0^s t^2(-j'(t))\,\mathrm{d}t$. For every $(j,g) \in \mathcal{M}_{\mathrm{rec}}(H)$, the shape scalars satisfy

$$K > 0, \qquad D > 0, \qquad Q_g = -\frac{K}{S} < 0. \tag{43}$$

(d) *Let $(j,g) \in \mathcal{M}_{\mathrm{rec}}(H)$, $\alpha,\psi \in \mathbb{R}$, $b,\rho \in \mathbb{R}^3$, $u,v \in \mathbb{R}^{3\times 3}$, and $\sigma \in \mathbb{C}^3$.*

(i) Every pair of distinct indices $i \neq k$ in $\{0,1,2\}$ is blind to ψ: for every continuous 2π-biperiodic observable $P : \mathbb{R}^2 \to \mathbb{R}$, $\langle P(\Phi_\psi(\varphi)_i, \Phi_\psi(\varphi)_k)\rangle_{\mathbb{T}^2} = \langle P(\Phi_0(\varphi)_i, \Phi_0(\varphi)_k)\rangle_{\mathbb{T}^2}$ is independent of ψ.

(ii) At $\mu = 0$, the triad coset average of the non-affine return angle reads $\cos\psi$ directly through K:

$$\big\langle \Delta\theta_{\mathrm{na}}\big(\alpha,0,F_{b,\rho}(\Phi_\psi(\varphi))\big)\big\rangle_{\mathbb{T}^2} = -\frac{3}{4}\,K\,(b_0b_1b_2)(\rho_0\rho_1\rho_2)\,\alpha^6\cos\psi. \tag{44}$$

(iii) Under independent uniform launch phases on $\mathbb{T}^3$ (for every fixed frequency triple, including exact resonance $\omega_0+\omega_1=\omega_2$), the leading ℓ-derivative $\mathcal{C}_3(A^{(0)},\delta A^{\mathrm{sh}})$ of the connected shifted-amplitude cumulant at $\ell = 0$ is set by the same scalar K with opposite sign:

$$\mathcal{C}_3\big(A^{(0)},\delta A^{\mathrm{sh}}\big) = \frac{K}{2S}\,(b_0b_1b_2)\big(\rho_0^2\rho_1^2+\rho_0^2\rho_2^2+\rho_1^2\rho_2^2\big)\,\alpha^5. \tag{45}$$

(iv) Replacing the shifted-energy ℓ-tangent δA^{sh} by the unshifted canonical ℓ-tangent δA^{un} cancels the leading connected derivative: $\mathcal{C}_3(A^{(0)},\delta A^{\mathrm{un}}) = 0$.

(e) *Let $\omega,k,I_* \in \mathbb{R}^r$ satisfy the resonance relation $\sum_{m=0}^{r-1}\omega_m k_m = 0$, let $t_b < 0$, and suppose $W,R_k : \mathbb{R}^r \to \mathbb{C}$ obey $\frac{\mathrm{d}}{\mathrm{d}t}W(I_*+tk) = -iR_k(I_*+tk)$ on $[t_b,0]$ with $t \mapsto R_k(I_*+tk)$ continuous and $W(I_*+t_bk) = 0$. Then the resonant action simplex stays invariant, $\sum_{m=0}^{r-1}\omega_m(I_{*,m}+tk_m) = \sum_{m=0}^{r-1}\omega_m I_{*,m}$ for all $t \in \mathbb{R}$, and $W(I_*) \neq 0$ implies $R_k(I_*+tk) \neq 0$ for some $t \in [t_b,0]$. Conversely, for every (j,g) obeying* (24) *with constant inertia $j(s) \equiv j_0 \in \mathbb{R}$, both leading channels go silent:*

$$K = 0, \qquad Q_g = 0, \qquad D = 0. \tag{46}$$

Proof. (a) Differentiating (33) along (37) using $\frac{\mathrm{d}}{\mathrm{d}s}|A_m|^2 = 2\,\mathrm{Re}(\overline{A_m}A_m') = -2b_m(\mathrm{Re}\,A_m)\frac{jp}{v}$ yields

$$\frac{\mathrm{d}E_{\mathrm{int}}}{\mathrm{d}s} = \frac{1}{2}j'p^2 + jp\left(\ell g' + \frac{-V'(q) + \sum_m b_m\,\mathrm{Re}\,A_m}{v}\right) + V'(q)\frac{jp}{v} - \sum_m b_m(\mathrm{Re}\,A_m)\frac{jp}{v} - \frac{\ell^2 J_{\mathrm{ax}}'}{2J_{\mathrm{ax}}^2},$$

in which both $-V'(q)jp/v$ and $(\sum_m b_m\,\mathrm{Re}\,A_m)jp/v$ cancel identically, giving (38). On the fixed energy surface $\frac{1}{2}v^2 + E_{\mathrm{int}}(s) = E = H_0 + (2\alpha^2)^{-1}$, the inverse speed is $\frac{\mathrm{d}t}{\mathrm{d}s} = v^{-1} = \alpha(1 - 2\alpha^2\Delta E_{\mathrm{int}}(s))^{-1/2} = \alpha + \alpha^3\Delta E_{\mathrm{int}}(s) + O(\alpha^5|\Delta E_{\mathrm{int}}|^2)$ for $\Delta E_{\mathrm{int}}(s) = E_{\mathrm{int}}(s) - H_0$, which is real-analytic in α near 0 uniformly on compact parameter sets (and rescales via $\hat{t} = t/\alpha$, $\hat{P}_s = \alpha P_s$, $\hat{p} = \alpha p$, $\hat{\ell} = \alpha\ell$ to unit reference speed $\hat{v} = \alpha v$). At $\ell = 0$ with $q(0) = 0$ and $p_c(0) = \alpha\mu$, $p(s) = \alpha(\mu + Fs) + O(\alpha^2)$ ($F = F_{b,\rho}(\varphi)$), and integrating (38) by parts gives $\Delta E_{\mathrm{int}}(s) = \frac{1}{2}j(0)\alpha^2\mu^2 + \int_0^s \frac{1}{2}j'(t)p(t)^2\,\mathrm{d}t = \alpha^2 P_\mu(s) + O(\alpha^3)$, whence $v^{-1} - \alpha = \alpha^5 P_\mu(s) + O(\alpha^6)$. Subtracting the affine reference system ($v^{-1} \equiv \alpha$ and $V'(q)$ replaced by $V''(0)q$, where $V'(q) - V''(0)q = O(q^3) = O(\alpha^6)$ enters p' first at $O(\alpha^7)$ because V is even) gives $p(s) - p_{\mathrm{aff}}(s) = \alpha^5 F\int_0^s P_\mu(t)\,\mathrm{d}t + O(\alpha^6)$ and

$$q'(s) - q'_{\mathrm{aff}}(s) = \alpha^6 j(s)\left((\mu + Fs)P_\mu(s) + F\int_0^s P_\mu(t)\,\mathrm{d}t\right) + O(\alpha^7);$$

integrating $\theta'(s) = g(s)q'(s)$ at $\ell = 0$ over $[0, S]$ yields the first expansion of (39). For canonical launch $q(0) = p_c(0) = 0$, evenness of V makes $(q, p_c, Y, Z, \varphi) \mapsto (-q, -p_c, -Y, -Z, \varphi + (\pi, \pi, \pi))$ an exact symmetry at $\ell = 0$, so $A_m(S, 0)$ reverses sign, $\langle A_m(S, 0)\rangle_{\mathbb{T}^3} = 0$, and $\kappa_3(0) = 0$. Differentiating in ℓ at $\ell = 0$ (where $p(0) = \ell g(0)$) gives $p_\ell(s) = g(s) + O(\alpha^2)$, $\partial_\ell\Delta E_{\mathrm{int}}|_0 = \alpha F L_g(s) + O(\alpha^2)$, $\partial_\ell(v^{-1})|_0 = \alpha^4 F L_g(s) + O(\alpha^5)$, and quadratic-in-$F$ derivative $\partial_\ell q'(s)|_0^{(2)} = \alpha^5 F^2 j(s)[\int_0^s L_g(t)\,\mathrm{d}t + sL_g(s) - \frac{1}{2}g(s)\chi(s)] + O(\alpha^6)$. Differentiating Duhamel's identity $e^{i\omega_m\alpha S}A_m(S, \ell) = A_m^{(0)}(\varphi)e^{-i\omega_m(t(S,\ell)-\alpha S)} - b_m\int_0^S e^{i\omega_m[\alpha S - t(S,\ell) + t(s,\ell)]}q'(s, \ell)\,\mathrm{d}s$ at $\ell = 0$ (using $t(S, 0) - \alpha S = O(\alpha^5)$ and $e^{i\omega_m\alpha s} = 1 + O(\alpha)$) yields $\partial_\ell(e^{i\omega_m\alpha S}A_m(S, \ell))\big|_{\ell=0} = \delta A_m^{\mathrm{sh}}(\varphi) + O(\alpha^6)$ with real $u_{mk}, v_{mk} = O(\alpha^4)$. Since $\tilde{A}_m(S, 0) = e^{-i\omega_m\alpha S}A_m^{(0)}(\varphi) + O(\alpha^2)$ and $\tilde{\delta}A_m^{\mathrm{sh}} = O(\alpha^4)$, Leibniz differentiation of $\kappa_3(\ell)$ at $\ell = 0$ gives $\partial_\ell\kappa_3(0) = \mathcal{C}_3(A^{(0)}, \delta A^{\mathrm{sh}}) + O(\alpha^6)$. Adding $b_m q(S, \ell) = b_m\int_0^S q'(s, \ell)\,\mathrm{d}s$ for $B_m = A_m + b_m q$ adds $+\alpha^5 b_m F^2 Q_g$ (up to a φ-independent shift removed by centering), cancelling the $O(\alpha^5)$ radiated term in $\tilde{\delta}A_m^{\mathrm{un}}$. Finally, shifting the section $s = 0$ to $s = \sigma(x)$ conjugates $W_{E,\ell}$ along flight time τ_σ, changing any additive one-turn cocycle h to $h + a \circ W_{E,\ell} - a$ [38] and leaving invariant cycle averages unchanged.

(b) For $0 < a < 1/2$, $D_a(\vartheta)^2 = \frac{1}{4} - a^2\cos^2\vartheta \geq \frac{1}{4} - a^2 = c > 0$ and $1 - 2a\cos\vartheta \geq 1 - 2a > 0$, so $u_a(\vartheta) > 0$. Since $\cos(\vartheta + \pi) = -\cos\vartheta$, $D_a(\vartheta + \pi) = D_a(\vartheta)$ and

$$u_a(\vartheta + \pi)\,u_a(\vartheta) = \frac{c(1 + 2a\cos\vartheta)(1 - 2a\cos\vartheta)}{4D_a(\vartheta)^2} = \frac{c(1 - 4a^2\cos^2\vartheta)}{4(\frac{1}{4} - a^2\cos^2\vartheta)} = c,$$

whilst $\sin(\vartheta + \pi) = -\sin\vartheta$ gives $w_a(\vartheta + \pi) = -w_a(\vartheta)$, proving (40). Adding and subtracting $u_a(\vartheta)$ and $c/u_a(\vartheta) = u_a(\vartheta + \pi)$ gives $u_a + c/u_a = \sqrt{c}/D_a$ and $(u_a - c/u_a)/(2(u_a + c/u_a)) =$

$-a\cos\vartheta$. Moreover,

$$w_a(\vartheta)^2 = \frac{a^2(1-\cos^2\vartheta)}{D_a(\vartheta)(D_a(\vartheta)+\sqrt{c})} = \frac{D_a(\vartheta)^2-c}{D_a(\vartheta)(D_a(\vartheta)+\sqrt{c})} = 1-\frac{\sqrt{c}}{D_a(\vartheta)} = 1-\left(u_a(\vartheta)+\frac{c}{u_a(\vartheta)}\right),$$

whence $\Psi(u_a,w_a) = u_a(1-u_a-w_a^2) = u_a(c/u_a) = c$. Direct differentiation yields $u_a'(\vartheta)/u_a(\vartheta) = \frac{a\sin\vartheta}{2D_a(\vartheta)^2}$. Whenever $\sin\vartheta \neq 0$, $u_a'(\vartheta)\neq 0$; whenever $\sin\vartheta = 0$, $\cos\vartheta = \pm 1 \neq 0$ and $w_a'(\vartheta) = \pm a/\sqrt{D_a(\vartheta)(D_a(\vartheta)+\sqrt{c})} \neq 0$. Hence $M_{\mathrm{rec}}(\vartheta) > 0$ is continuous, π-periodic, and even about $\pi/2$. Consequently $s_{\mathrm{rec}}'(\vartheta) = \sqrt{M_{\mathrm{rec}}(\vartheta)} > 0$ everywhere, $s_{\mathrm{rec}}(\vartheta+\pi) = s_{\mathrm{rec}}(\vartheta)+H$, $s_{\mathrm{rec}}(\pi-\vartheta) = H - s_{\mathrm{rec}}(\vartheta)$, and $s_{\mathrm{rec}}(k\pi) = kH$ ($k\in\mathbb{Z}$), so $s_{\mathrm{rec}}:\mathbb{R}\to\mathbb{R}$ is a strictly increasing global diffeomorphism. Since $\vartheta(s)$ maps $[0,H/2]$ strictly increasingly onto $[0,\pi/2]$, $\cos\vartheta(s)$ is strictly decreasing and nonnegative on $[0,H/2]$; hence $D_a(\vartheta(s))$ is strictly increasing, $j(s) = \frac{\sqrt{c}}{m_p R^2 c D_a(\vartheta(s))}$ is strictly decreasing on $[0,H/2]$, and $g(s) = -a\cos\vartheta(s)$ is strictly increasing on $[0,H/2]$ with $g(0) = -a < 0$, placing $(j,g)\in\mathcal{M}_{\mathrm{rec}}(H)$. Under $\lambda = \sqrt{m_p}\,R$ at fixed $\eta = H_{\mathrm{cav}}/R$, $M_{\mathrm{rec}}\propto\lambda^2$, $s_{\mathrm{rec}}\propto\lambda$, $j\propto\lambda^{-2}$, $g\propto\lambda^0$, and $\chi\propto\lambda^0$, so $K\propto\lambda^{-2}\cdot\lambda\cdot\lambda = \lambda^0$ is invariant.

(c) Write $C(s) = \int_0^s j(t)\,\mathrm{d}t$, $B(s) = \int_0^s t\,j(t)\,\mathrm{d}t$, and $M_2(s) = \int_0^s t^2 j(t)\,\mathrm{d}t$. When $j\in C^1$, $\frac{\mathrm{d}}{\mathrm{d}s}\int_0^s t^2(-j'(t))\,\mathrm{d}t = -s^2 j'(s) = \frac{\mathrm{d}}{\mathrm{d}s}(-s^2 j(s)+2B(s))$, giving $\chi(s) = \int_0^s t^2(-j'(t))\,\mathrm{d}t$. Under (24), $C(s+H) = C(s)+C(H)$, $B(s+H) = B(s)+HC(s)+B(H)$ with $B(H) = \frac{1}{2}HC(H)$, and setting $R(s) = j(s)g(s)$ ($R(s+H) = -R(s)$, $R(H-s) = -R(s)$) makes $\int_0^S R(s)j(s)\,\mathrm{d}s = \int_0^S sR(s)j(s)\,\mathrm{d}s = \int_0^S R(s)C(s)\,\mathrm{d}s = 0$. Integrating Q_g in (27) once by parts on $[0,S]$ using $C'(s) = j(s)$ and $B'(s) = s\,j(s)$ expresses $-SQ_g$ and K as $\int_0^S R(s)\mathcal{P}(s)\,\mathrm{d}s$ and $\int_0^S R(s)\mathcal{K}(s)\,\mathrm{d}s$. Splitting $\int_0^S = \int_0^H + \int_H^{2H}$ via $R(s+H) = -R(s)$ and folding $[0,H]$ onto $z\in[0,H/2]$ via $s = \frac{H}{2}\mp z$ and $R(\frac{H}{2}+z) = -R(\frac{H}{2}-z)$ collapses all boundary moments identically onto the common folded integral

$$\begin{aligned} K = -SQ_g = 4H^2\int_0^{H/2} &\big(-j(\tfrac{H}{2}-z)g(\tfrac{H}{2}-z)\big) \\ &\times\left[\int_0^z \big(j(\tfrac{H}{2}-z)-j(\tfrac{H}{2}-u)\big)\,\mathrm{d}u + 2z\big(j(\tfrac{H}{2}-z)-\bar{j}\big)\right]\mathrm{d}z, \end{aligned} \tag{47}$$

where $\bar{j} = C(H)/H = \frac{2}{H}\int_0^{H/2} j(\frac{H}{2}-u)\,\mathrm{d}u$, proving $Q_g = -K/S$. Similarly, folding (28) yields

$$D = 2\int_0^{H/2} j(t)\big(6t^2-6Ht+H^2\big)\,\mathrm{d}t. \tag{48}$$

Expanding $P_\mu(s) = \frac{1}{2}\mu^2 j(s) + \mu F(s\,j(s)-C(s)) - \frac{1}{2}F^2\chi(s)$ and substituting into (30), the μ^3 and $\mu^2 F$ integrals vanish by the three parity identities above, while the μF^2 and F^3 coefficients equal $Q_g = -K/S$ and $-\frac{1}{2}K$ respectively, giving (42). Now assume $(j,g)\in\mathcal{M}_{\mathrm{rec}}(H)$ and put $j_*(z) = j(\frac{H}{2}-z) > 0$ and $r_*(z) = -j(\frac{H}{2}-z)g(\frac{H}{2}-z)$ on $[0,H/2]$. Because j is strictly decreasing and g is strictly increasing on $[0,H/2]$ with $g(H/2) = 0$, both j_* and $-g(\frac{H}{2}-\cdot)$ are strictly increasing on $[0,H/2]$, with $r_*(z) > 0$ on $(0,H/2]$. Hence $\int_0^z (j_*(z)-j_*(u))\,\mathrm{d}u > 0$ on $(0,H/2]$, so the first part of (47) is strictly positive. Moreover, for any

continuous strictly increasing $w : [a, b] \to \mathbb{R}$ and continuous strictly increasing $u : [a, b] \to \mathbb{R}$ with $\int_a^b u(z)\,\mathrm{d}z = 0$ ($a < b$), the Intermediate Value Theorem gives a unique root $z_0 \in (a, b)$ of u, so $(w(z)-w(z_0))u(z) > 0$ on $(a, b)\setminus\{z_0\}$ and $\int_a^b w(z)u(z)\,\mathrm{d}z = \int_a^b (w(z)-w(z_0))u(z)\,\mathrm{d}z > 0$. Applying this zero-mean Chebyshev inequality on $[0, H/2]$ to the strictly increasing product $w(z) = 2z\,r_*(z)$ and the zero-mean strictly increasing deviation $u(z) = j_*(z) - \bar{\jmath}$ proves $\int_0^{H/2} 2z\,r_*(z)(j_*(z) - \bar{\jmath})\,\mathrm{d}z > 0$, whence $K > 0$ and $Q_g = -K/S < 0$. Applying the same inequality in (48) to the strictly increasing functions $w(t) = -j(t)$ and $u(t) = -(6t^2 - 6Ht + H^2)$ on $[0, H/2]$ (where $\int_0^{H/2} u(t)\,\mathrm{d}t = -[2t^3 - 3Ht^2 + H^2t]_0^{H/2} = 0$) yields $D > 0$, completing (43).

(d) For (i), when $(i, k) = (0, 1)$ or $(1, 0)$, $\Phi_\psi(\varphi)_i$ and $\Phi_\psi(\varphi)_k$ do not contain ψ. When $(i, k) = (0, 2)$ or $(2, 0)$, for each fixed φ_1 the inner integral over $\varphi_2 \in [0, 2\pi]$ shifts $\varphi_2 \mapsto \varphi_2 + \varphi_1 + \psi$ by 2π-periodicity of P to become independent of ψ. When $(i, k) = (1, 2)$ or $(2, 1)$, setting $Q(x, y) = P(y, x + y)$ (or $P(x + y, y)$) makes $G(x) = \int_0^{2\pi} Q(x, y)\,\mathrm{d}y$ continuous and 2π-periodic in x, so $\int_0^{2\pi} G(\varphi_1 + \psi)\,\mathrm{d}\varphi_1 = \int_0^{2\pi} G(\varphi_1)\,\mathrm{d}\varphi_1$. For (ii), setting $\mu = 0$ in (42) gives $\Delta\theta_{\mathrm{na}}(\alpha, 0, F) = -\frac{1}{2}\alpha^6 K F^3$. Writing $r_m = b_m\rho_m$ and $F_{b,\rho}(\Phi_\psi(\varphi)) = r_0\cos\varphi_1 + (r_1 + r_2\cos(\varphi_1 + \psi))\cos\varphi_2 - r_2\sin(\varphi_1 + \psi)\sin\varphi_2$, integrating the cube over $\varphi_2 \in [0, 2\pi]$ via $\int_0^{2\pi}(A + B_1\cos\varphi_2 + B_2\sin\varphi_2)^3\,\mathrm{d}\varphi_2 = 2\pi A^3 + 3\pi A(B_1^2 + B_2^2)$ leaves

$$2\pi r_0^3\cos^3\varphi_1 + 3\pi r_0\cos\varphi_1\big(r_1^2 + r_2^2 + 2r_1r_2\cos(\varphi_1 + \psi)\big).$$

Integrating over $\varphi_1 \in [0, 2\pi]$ annihilates $\cos^3\varphi_1$, $\cos\varphi_1$, and $\cos\varphi_1\sin\varphi_1$, while

$$\int_0^{2\pi}\cos^2\varphi_1\,\mathrm{d}\varphi_1 = \pi$$

contributes $6\pi^2 r_0 r_1 r_2\cos\psi$; dividing by $(2\pi)^2$ and multiplying by $-\frac{1}{2}\alpha^6 K$ yields (44). For (iii) and (iv), write $A_m^{(0)}(\varphi) = \rho_m(\cos\varphi_m - i\sin\varphi_m)$. On $[0, 2\pi]$,

$$\int_0^{2\pi} A_m^{(0)}\,\mathrm{d}\varphi_m = \int_0^{2\pi}\overline{A_m^{(0)}}\,\mathrm{d}\varphi_m = 0,$$
$$\int_0^{2\pi}(\cos\varphi_m)A_m^{(0)}\,\mathrm{d}\varphi_m = \int_0^{2\pi}(\cos\varphi_m)\overline{A_m^{(0)}}\,\mathrm{d}\varphi_m = \pi\rho_m.$$

Expanding the first slot $\tilde{\delta}A_0^{\mathrm{sh}}\,A_1^{(0)}\overline{A_2^{(0)}}$ into eight separable products on $\mathbb{T}^3$, every linear phase-jitter term, every constant shift $\sigma_0 - \langle\delta A_0^{\mathrm{sh}}\rangle_{\mathbb{T}^3}$, and every term of $F_{b,\rho}(\varphi)^2$ other than $2r_1r_2\cos\varphi_1\cos\varphi_2$ leaves at least one of φ_1 or φ_2 carrying a pure $A_m^{(0)}$ or $\overline{A_m^{(0)}}$ factor of zero integral. The sole surviving product in slot 0 has $\mathbb{T}^3$-average

$$\frac{1}{(2\pi)^3}(2\pi)\big(-2\alpha^5 b_0 Q_g r_1 r_2\big)(\pi\rho_1)(\pi\rho_2) = -\frac{1}{2}\alpha^5 Q_g(b_0b_1b_2)\rho_1^2\rho_2^2.$$

Slots 1 and 2 contribute $-\frac{1}{2}\alpha^5 Q_g(b_0b_1b_2)\rho_0^2\rho_2^2$ and $-\frac{1}{2}\alpha^5 Q_g(b_0b_1b_2)\rho_0^2\rho_1^2$; substituting $-Q_g = K/S$ yields (45). In δA_m^{un}, the $F_{b,\rho}(\varphi)^2$ term cancels identically before averaging, giving $\mathcal{C}_3(A^{(0)}, \delta A^{\mathrm{un}}) = 0$.

(e) Linearity gives $\sum_m \omega_m(I_{*,m} + tk_m) = \sum_m \omega_m I_{*,m} + t\sum_m \omega_m k_m = \sum_m \omega_m I_{*,m}$. If $R_k(I_* + tk) = 0$ for all $t \in [t_b, 0]$, the Fundamental Theorem of Calculus gives $W(I_*) - W(I_* + t_b k) = \int_{t_b}^{0} (-iR_k(I_* + tk))\,\mathrm{d}t = 0$, so $W(I_*) = W(I_* + t_b k) = 0$, a contradiction when $W(I_*) \neq 0$. Finally, when $j(s) \equiv j_0$, $\chi(s) = -s^2 j_0 + 2j_0(\frac{1}{2}s^2) = 0$ identically, whence $K = 0$, $D = 0$, and $Q_g = -K/S = 0$ by (42). □

8 The field holds the program

A computer holds its program as a state. Lovelace wrote that the Analytical Engine "weaves algebraical patterns just as the Jacquard-loom weaves flowers and leaves" [42]: the pattern lives in the cards, the loom stays as it was built, and in the junction the card is the field. The Kelvin bias ΔP_B of Theorem A10 is the pressure the field sets (Theorem A6); that bias alone chooses the gate, so one fixed network of junctions runs every program held in the field.

Theorem A13 (One network, every function)**.** *Let every junction be that of Theorem A10, its output restored to the pressure $P_0 + y\Delta P$ of its bit y as in Theorem A10(d), with bias $\theta = P_c - \Delta P_B$.*

(a) A program junction, its inputs held at P_0 and $P_0 + \Delta P$, settles at $P_j - P_0 = \frac{1}{3}\Delta P$ and reads 1 when θ lies in the NAND *window* (21) *and* 0 *when θ lies in the* NOR *window* (22)*.*

(b) For $n \geq 1$ the network U_n has fixed wiring: program junctions p_v, one for each $v \in \{0,1\}^n$, and $2^n - 1$ multiplexers

$$\mathrm{MUX}(s, x_0, x_1) = \mathrm{NAND}\big(\mathrm{NAND}(x_0, \mathrm{NAND}(s, s)), \mathrm{NAND}(x_1, s)\big) \tag{49}$$

of four junctions held at one bias in the NAND *window. The multiplexer of U_1 selects with a_1 between p_0 and p_1, and the root of U_n selects with a_n between copies of U_{n-1} on the p_v with $v_n = 0$ and on those with $v_n = 1$. For every assignment β of biases to the program junctions and every input $a \in \{0,1\}^n$, $U_n(\beta, a) = p_a(\beta)$.*

(c) Every $f : \{0,1\}^n \to \{0,1\}$ factors through U_n: under the preparation $\beta(f)$ that places p_v in the NAND *window when $f(v) = 1$ and in the* NOR *window when $f(v) = 0$, $U_n(\beta(f), a) = f(a)$ for every $a \in \{0,1\}^n$, and every map $\{0,1\}^n \to \{0,1\}^m$ factors through m copies of U_n.*

Proof. (a) By (20) at $(a, b) = (0, 1)$ the detector reads $[\frac{1}{3}\Delta P < \theta]$, which is 1 on (21) and 0 on (22). (b) Each junction of a multiplexer computes $1 - xy$ by Theorem A10(b), and restored outputs compose, so with $\mathrm{NAND}(s, s) = 1 - s$ the multiplexer is $1 - \big(1 - x_0(1 - s)\big)\big(1 - x_1 s\big)$, which is x_0 at $s = 0$ and x_1 at $s = 1$. This settles $n = 1$. For $n > 1$ write $a = (a', a_n)$; by induction the two copies of U_{n-1} read $p_{(a',0)}$ and $p_{(a',1)}$, and the root selects $p_{(a',a_n)} = p_a$.

(c) By (a), $p_v(\beta(f)) = f(v)$ for every v, and (b) gives $U_n(\beta(f), a) = f(a)$; a map to $\{0,1\}^m$ prepares one copy to each coordinate. □

In (1), the field and the input pressures form the state of U_n, giving $f = D \circ \Phi \circ E_f$: the wiring, the settling Φ and the readout D never change, while the preparation E_f writes $\beta(f)$ into the field for each f. Every circuit of the junctions of Theorems A10 and A11 computes a Boolean function and runs on U_n prepared to its truth table.

9 Conclusion

The vessel holds one circulation at every viscosity, its midplane split at the ring of least pressure (Theorems A1–A5). At uniform susceptibility the field sets the pressure (Theorem A6), crossed with heat it turns the water, and the centrifugal curl selects the swirl the circulation keeps (Theorems A7 and A8). Heat drives the circulation and the vessel holds it to be read, Schopenhauer's world as will and as representation: a blind, restless striving and the form in which it appears [57]. Twelve vessels on a ring carry seven levels (Theorem A9), and the twelve is a three and a four (Theorem A0(d)). The junction of three waters keeps a trit (Proposition 5.1, Theorem A10), one bias chooses the gate that reads it (Theorem A11), and one network with fixed wiring computes every Boolean function, its program held in the field (Theorem A13). All of (4) composes into one factorization, the evolution of a geometry contracted through what its observation keeps, bringing eleven earlier papers under Theorem A0 and Proposition A.1 (Table 2).

Aristotle asked why matter is some definite thing, and answered that "what we seek is the cause, i.e. the form" [1]. In the turn, that form is the variation of the inertia, fixing the single constant K that stands in front of both the angle reading a relation among three phases and the shifted amplitudes into which a relation among three modes is written, going silent at constant inertia (Theorem A12). The two waves of the water-bearer's glyph are the read and the write, Schleiermacher's hermeneutic circle in which the part is understood through the whole and the whole through the part [55]. The relation among the phases was there before the water went around, and the turn returned it as an angle. Between a circle that closes completely and something immediate kept outside it, the turn answers with a third: the ternary, the trit no bit holds and the relation among three phases that no pair carries.

Observation-relative contraction grows out of [39, 40] alongside the classical quotients of Section 1. The inviscid spherical case $R = H$ is Hill's vortex [18, 25, 4], extended to viscous spheroids by O'Brien [48] and surveyed among exact Navier–Stokes solutions in [59] and [12]. The Kelvin force and thermomagnetic convection stand on [52, 14], secondary meridional flow from height-varying swirl on [10], circulant spectra on [11], dihedral rings on [15], and damped second-order resonance on [37]. The split of the twelve is the Chinese remainder theorem in unit-fraction form [40], linked to Mordell's congruences [45]; two-sheet parity comes from [41]; and kill, propagate and generate are the carry states of carry look-ahead [60] under the prefix

operator of [6].

Triad resonance in fluids is classical [9], bispectra capture three-phase relations invisible to pairs [16, 24], oscillator-bath coupling follows [61], return maps and symmetry reduction rest on [43, 22], and geometric phases in chaotic Hamiltonian systems are developed in [58]. The table of modes and elements comes down from the astrologers, its four elements matching Aristotle's pairs of qualities [3]; fluidic logic has been realized in the laboratory [49], and Turing-complete steady Euler flows exist in three dimensions [8].

The Aquarius hypothesis

Theorem A13 proves the computer. One fixed network of junctions computes every Boolean function, its program held in the field. What stays open is a physical claim about a real fluid: a closed reciprocal circulation is a channel that reads and writes collective phase information, with coupling constant K fixed by the shape alone (Theorem A12). Returns compose (Theorem A0), and under this hypothesis the channel composes with them, so a computation becomes a composition of returns where the retained state is the phase relation and the heat left by one move drives the next. Theorem A12(b) proves that the paired fluid geometry lies in the reciprocal class, and the triad law hands the hypothesis its first measurable test: an angular shift proportional to $K\, b_0 b_1 b_2\, \rho_0 \rho_1 \rho_2 \cos\psi$, a prefactor of opposite sign in the shifted-amplitude connected response, and silence of both leading channels at constant inertia. Jacobi held that a chain of conditions, however complete its demonstration, rests on something given immediately, outside the chain [19]; here the proofs hand the verdict to the real water. The experiment decides it.

10 Executable and visual verification

Independently of Lean, `verify.py` checks the finite and numerical content directly, using exact rational arithmetic for algebraic identities and repeating the integrations with SciPy where installed.

1. *Exact arithmetic.* It checks the solenoidal, boundary and Ψ-orthogonality identities of Theorems A1 and A5 on 256 rational instances together with the quarter fluxes $\pm 1/4$ of Theorem A2; the momentum balance, vorticity stretching, harmonic vorticity and midplane minimum of Theorem A4 on 256 rational 8-tuples $(R, H, \rho, U, \mu, x, y, z)$; the cross-gradient curl, Carnot inequalities, Lorentz orthogonality and centrifugal curl of Theorems A6 to A8, and the torsion that holds the swirl of Theorem A5 on 256 rational 10-tuples $(R, H, \rho, U, \mu, \omega, B_0, x, y, z)$; the dihedral group of order 24 and the circulant eigenmodes at $N = 12$ of Theorem A9; and the maps `unify`, `tritShift` and `liftUnifier` on all 27 triples of Trit, the $\mathfrak{su}(3)$ commutators, the traceless shear tensors, the inversion of $\mathbb{Z}/12\mathbb{Z} \cong \mathsf{Trit} \times \mathbb{Z}/4\mathbb{Z}$ with its sharp dial bounds $1/6$, $1/8$ and $1/24$, and the sharp endpoints of the half-open NAND and NOR windows (Theorems A0, A10

and A11). For Theorem A13 it runs the program junction, the multiplexer (49) on all eight inputs, and U_n with its $5 \cdot 2^n - 4$ junctions on every Boolean function of up to three inputs and 256 random functions each of four and five inputs, at interior biases and at the closed ends of both windows.

2. *Numerical integration.* On the shells $a \in \{0.05, 0.15, 0.25, 0.35, 0.45, 0.49\}$, the elliptic periods $P(a)$ computed by the arithmetic–geometric mean and by quadrature agree with a three-dimensional integration of $\mathbf{F}_\omega$ within 10^{-8}, and a 24-dimensional integration of the second-order ring of twelve agrees with the modal semigroup (19) within 10^{-10}. For Theorem A12, on the paired shells $a \in \{0.15, 0.30, 0.42\}$, quadrature on $[0, S]$ confirms $\Psi(u_a, w_a) = c$, the half-turn exchange $u_a(\vartheta + \pi) = c/u_a(\vartheta)$ and $w_a(\vartheta + \pi) = -w_a(\vartheta)$, and $K > 0$ and $D > 0$. It confirms $Q_g = -K/S$ and the invariance of K when m_p and (R, H_{cav}) are rescaled at fixed H_{cav}/R within 10^{-11}; pairwise blindness to ψ and the triad read (44), by quadrature on the two-torus, within 10^{-12}; the shifted-amplitude write (45) and the unshifted cancellation $\mathcal{C}_3(A^{(0)}, \delta A^{\mathrm{un}}) = 0$, by quadrature on the three-torus, within 10^{-12}; and $K = Q_g = D = 0$ at constant inertia within 10^{-14}.

3. *The plates.* The two plates published with this paper, the ring of twelve vessels `figures/00-cover-4k.png` and the meridional section `figures/01-section-4k.png`, are built by `visual/build_scene.py` and `visual/build_section.py`, which verify every vertex of the streamlines, equal-area rings, walls, pressure well and $\mathbb{Z}/3\mathbb{Z}$-symmetric ring against the exact formulas within 10^{-12} in double precision before Blender renders them, and check the streamline, ring and wall curves that Blender stores in single precision within 5×10^{-6}. The script `visual/build_orbit.py` checks the loop closure of the camera path for the orbit animation `figures/Aquarius_Orbit.mp4`, 288 frames over 12.0 s at 24 fps through a turn of $2\pi/3$, within 10^{-12}.

Run `python3 verify.py`; it exits with a nonzero status on any failure and prints `PASS` with a summary otherwise. These executable checks stand beside the proofs.

11 Formal verification

The theorems and propositions of this paper are machine-checked in Lean 4.35.0 [46] with Mathlib v4.35.0 [44] (`56a00585...`). The text proves their analytic parts: the elliptic-integral form (9) of the period in Theorem A3, the rectified angle of Theorem A5, the scale invariance of K in Theorem A12(b), the uniform remainder bounds and the section transport of Theorem A12(a), and the one-line corollaries (pressure absorption in Theorem A6, the order 24 and half-angle form in Theorem A9(b), and the largest margin in Theorem A10(b)). Table 1 lists the declarations and their files in `lean/`; `lake build Aquarius` rebuilds all eleven submodules, and `lake env lean Aquarius.lean` prints the axioms of all 91 listed declarations. None uses `sorry`, `admit`, a custom axiom or `native_decide`, and `#print axioms` reports only `propext`, `Classical.choice` and `Quot.sound`, or no axiom.

Table 1: Machine-checked Lean 4 declarations in `lean/Aquarius/`.

Statement	Checked Lean declarations	Module
Mirror parity	`mirror_parity_decomp`, `mirror_parity_unique`, `mirrorOdd_midplane_zero`	`Basic.lean`
Theorem A0	`A0_quotient_map_iff`, `A0_quotient_map_unique`, `A0_reduced_semigroup`, `A0_iter_contraction`, `A0_contracted_factorization`, `A0_quotient_card_le`, `A0_exact_boundary_equiv_unique`, `A0_mordell_12_int`, `mordell12Equiv`, `A0_dial_rounding_exact`	`StateContraction.lean`
Prop. A.1	`A0_fiber_seed_bound`, `A0_threshold_chain_bound`, `A0_cut_coboundary_and_linear_kernel`, `A0_irreducible_incompressibility`	`StateContraction.lean`
Theorem A1	`A1_divergence_zero`, `A1_no_boundary_flux`, `A1_psi_invariant`, `A1_mirror_flux`	`Circulation.lean`
Theorem A2	`A2_split_unique`, `A2_equal_area_split`, `A2_integrated_split`, `A2_dimensional_transfers`, `A2_quadratic_profile_unique`	`Circulation.lean`
Theorem A3	`A3_orbit_circle`, `A3_orbit_physical_invariant`, `A3_orbit_interior`, `A3_phase_ode_trajectory`, `A3_orbit_periodic`, `A3_orbit_mirror`	`Circulation.lean`
Theorem A4	`A4_curl_velocity_eq_vorticity`, `A4_laplacian_velocity`, `A4_euler_closure`, `A4_navier_stokes_closure`, `A4_midplane_pressure_strict_min`, `A4_vorticity_stretching_and_harmonic`	`EulerClosure.lean`
Theorem A5	`A5_swirl_divergence_zero`, `A5_swirl_boundary_and_psi`, `A5_spiral_slope_mirror_odd`, `A5_return_closure_iff`	`MagnetoCoupling.lean`
Theorem A6	`A6_kelvin_curl_zero_of_const_chi`	`MagnetoCoupling.lean`
Theorem A7	`A7_kelvin_cross_gradient`, `A7_axisymmetric_azimuthal_curl`, `A7_thermal_affinity_and_carnot`	`MagnetoCoupling.lean`
Theorem A8	`A8_lorentz_kinematic_orthogonality`, `A8_lorentz_divergence_zero`, `A8_lorentz_mirror_parity`, `A8_centrifugal_curl`, `A8_centrifugal_curl_zero_iff`, `A8_mirror_odd_spin_nonconservative`	`MagnetoCoupling.lean`
Theorem A9	`A9_circulant_eigenvector`, `A9_zetaRoot_pow_N`, `A9_dihedral12_relations`, `A9_restoring12_pos`, `A9_ring12_seven_levels`, `A9_first_order_relaxation_ode`, `A9_modalFlow_semigroup`, `A9_modalFlow_ode_and_energy`, `A9_damped_dissipation_and_resonance`	`FourierRing.lean`

Statement	Checked Lean declarations	Module
Prop. 5.1	`tritShift_assoc`, `unify_assoc`, `liftUnifier_left_inv`, `liftUnifier_right_inv`, `cyclic_two_swaps`, `ternary_no_bool_injection`, `matrix_shift_unitary`, `channel_unify_compose`, `su3_bracket_closure`, `shear_tensor_decomposition`	`TernaryAlgebra.lean`
Theorem A10	`A10_junction_solution`, `A10_trit_bijection`, `A10_biased_nand_iff`, `A10_nand_endpoint_sharpness`, `A10_nand_bilateral_margin_iff`, `A10_biased_nor_iff`, `A10_four_nand_xor_and_carry_recovery`	`Junction.lean`
Theorem A11	`A11_continuum_torus_contraction`, `A11_ring12_spectral_and_trit_synthesis`, `A11_master_factorization_and_reconfiguration`	`Synthesis.lean`
Theorem A12	`A12_arc_energy_exchange_ode`, `A12_paired_geometry_instantiates`, `A12_common_scalar_and_positivity`, `A12_triad_read_and_connected_response`, `A12_simplex_duality_and_metric_control`	`ReciprocalResponse.lean`
Theorem A13	`A13_program_junction`, `A13_mux_select`, `A13_network_reads_program`, `A13_every_function_factors`, `A13_every_map_factors`	`Program.lean`

A The sixteen earlier papers

The main theorems of eleven of the sixteen earlier papers are instances of Theorem A0 under the following bridge laws.

Proposition A.1 (Bridge laws and the converse)**.** *(a) For any map $F : \mathcal{G} \to \mathcal{Y}$, target $y \in \mathcal{Y}$, and seed map $b : \mathcal{G} \to B$ injective on $F^{-1}(y)$, the restriction $b|_{F^{-1}(y)}$ forms a retained boundary for the identity on $F^{-1}(y)$, and $|F^{-1}(y)| \le |B|$ whenever both sets are finite.*

(b) Given $m \in \mathbb{N}$ integer thresholds $\theta_0, \dots, \theta_{m-1} \in \mathbb{Z}$, the indicator vector $C(u) = ([u \ge \theta_0], \dots, [u \ge \theta_{m-1}]) \in \{0,1\}^m$ $(u \in \mathbb{Z})$ is componentwise monotone in u and attains at most $m+1$ distinct values on any finite prefix set $S \subset \mathbb{Z}$.

(c) For any cycle $F = S_k \circ P_k : \mathcal{G} \to \mathcal{G}$ with additive prefix and suffix costs $C(x) = C_k(x) + B_k(P_k(x))$ and rotated cut total $T_k(x) = B_k(P_k(x)) + C_k(S_k(P_k(x)))$ in an abelian group, the cut defect obeys the coboundary law $T_k(x) - C(x) = C_k(F(x)) - C_k(x)$. Over a ring K, for modules V, W, U, linear maps $L : V \to W$ and $S : V \to U$, and offset $a \in W$, S is a retained boundary for $x \mapsto Lx + a$ if and only if $\ker S \subseteq \ker L$.

(d) As an incompressibility converse, let $(A_i)_{i \in I}$ be linear endomorphisms of a K-module V whose only common invariant submodules are $\{0\}$ and V. If $Q : V \to U$ is linear, $c \in U$, and arbitrary maps $g_i : U \to U$ satisfy $Q(A_i x) + c = g_i(Qx + c)$ for all $i \in I$ and $x \in V$, then either $Q = 0$ or Q is injective.

Proof. (a) If $x_1, x_2 \in F^{-1}(y)$ satisfy $b(x_1) = b(x_2)$, injectivity on the fiber gives $x_1 = x_2$, establishing the retained-boundary condition and $|F^{-1}(y)| \le |B|$.

(b) If $u \le v$ and $\theta_j \le u$, transitivity gives $\theta_j \le v$. Map each $w \in C(S)$ to its ones count $|I_w| \in \{0, \dots, m\}$, where $I_w = \{j : w_j = 1\}$. For $u, v \in S$ with $|I_{C(u)}| = |I_{C(v)}|$, assuming $u \le v$ (or symmetrically $v \le u$) gives $I_{C(u)} \subseteq I_{C(v)}$ with equal finite cardinality, whence $I_{C(u)} = I_{C(v)}$ and $C(u) = C(v)$. Thus $w \mapsto |I_w|$ injects $C(S)$ into $\{0, \dots, m\}$, so $|C(S)| \le m + 1$.

(c) Subtracting $C(x) = C_k(x) + B_k(P_k(x))$ from $T_k(x) = B_k(P_k(x)) + C_k(F(x))$ cancels $B_k(P_k(x))$. If S is a retained boundary for $x \mapsto Lx + a$ and $v \in \ker S$, then $Sv = 0 = S0$ implies $Lv + a = a$, so $Lv = 0$. Conversely, if $\ker S \subseteq \ker L$ and $Sx = Sy$, then $x - y \in \ker S \subseteq \ker L$, giving $Lx = Ly$ and $Lx + a = Ly + a$.

(d) Evaluating at $x = 0$ gives $c = g_i(c)$. For $x \in \ker Q$, $Q(A_i x) + c = g_i(c) = c$, so $A_i x \in \ker Q$. Common invariance forces $\ker Q \in \{\{0\}, V\}$, whence Q is injective or identically zero. □

Table 2: Direct instances of Theorem A0 and Proposition A.1 among the sixteen prior papers.

Paper and theorem	Carrier $\mathcal{G}$	Observation $\mathcal{O}$	Boundary $b : \mathcal{G} \to B$	Size or rank
Two-Stage Addition [31, Thms. 1, 7]	$\{0, \dots, B-1\}^3$	$Q_{C,S,H}$ on $\{0,1\}^2$	Normal-form label $\ell(v,t)$	$5B-4$ (exact)
Cascaded Sum-and-Carry [32, Thms. 1, 5]	$\{0, \dots, B-1\}^{k+1}$	$Q_{\mathbf{x}}$ on $\{0,1\}^k$	Carry-profile ranker	$B^{k-t}((B-1)A_t+1)$ (exact)
Ordered Carry Frontiers [34, Thm. 1]	$\{0, \dots, B^i-1\}$ $(i<w)$	Upper-digit residuals $R_{i,u}$	Threshold carry chain $C_i(u)$	$1+d_i \le m+1$ (exact)
Three-Bit Cyclic Inversion [27, Thms. 1, 2]	Fiber $F_{c,k}^{-1}(y)$ $(w \ge 4)$	Preimage word x	Seed $(x_0,x_1,x_2) \in \{0,1\}^3$	$\lvert B\rvert = 8$ (retained, sharp)
Bounded Tap Span [29, Lem. 4, Thm. 1]	Valid prefixes $(j \ge h)$	Legal suffixes in $\mathcal{A}(D,S)$	Frontier $(a_{j-h..j-1}, t_{j-h})$	2^{h+1} per seed, layer
Affine Carry Inversion [30, Thm. 1, Prop. 3]	Fiber $\mathcal{A}_{D,S}(L) \subseteq \mathbb{F}_2^w$	Nonlinear carry signature	XOR signature $\delta \in \Delta_D$	$\le F_{w+1}$ affine cells
Exact State Contraction [39, Thm. 1, Prop. 1]	System states S	Finite-word traces on U^*	Intertwining quotient $\pi : S \twoheadrightarrow Q$	$\lvert Q\rvert$ (trace-exact)
The Unifier [40, T4, T8–T16, T20]	Carry windows; dial $\mathbb{Z}/12\mathbb{Z}$	$(F_S(0), F_S(1))$; tick x	Trit $F_S(1)+F_S(0)$; split μ_{12}	3; 3×4 (exact)
Affine Cycles [38, Lem. 1, Thms. 5, 8]	Initial states $\mathbb{R}^d$	Cut defects $R(M-I)x + Rb$	Row-space map $S : \mathbb{R}^d \twoheadrightarrow \mathbb{R}^q$	$\operatorname{rank}(R(M-I))$ (exact)
Damped Recurrences [37, Lem. 6, Thm. 7]	Initial pairs $(x_0,x_1) \in \mathbb{R}^2$	Infinite peak $\sup_{n\ge 0}\lvert x_n\rvert$	Sample prefix $(x_0, \dots, x_{N_*-1})$	$N_* \in \{2,3\}$ on hexagon
Lattice-Phase Selection [36, Thms. 1, 2, 4]	Lattice origin $\theta \in [0, \Delta)$	Feasible set and grid indices k_i	Residue cell of $\mathcal{B}$	$\le 2D+1$ phase cells

Five more papers stand in a named relation to Theorem A0 alongside the eleven direct instances. *Cyclic-Shift Energies* [35] is a dual: its second section yields an exact linear boundary under Proposition A.1(c) for $\mathcal{O}(c) = \|Lc\|_A^2$ on $\mathbb{R}^{NP}$, with $q_{\min} = \operatorname{rank}(L^\top AL) = \sum_{k=0}^{P-1} \operatorname{rank} G_k$ and a retained potential of $2P$ entries (Theorems 1 and 3 there), while its main theorem proves a cyclotomic field-norm gap for integer occupancies at prime periods (Theorem 6 there). *Winding and Mirrored* [41] supplies the geometry carried into a fluid here: the two-sheet Hamiltonian ring (Theorems W1 to W4), the split of a field into torsion, drive and mean (W5), the seam dipole $\pm q$ (W6), the dihedral group of order 24 (W8) and the double spiral (W11); its count of $(p^2-1)/4$ sign-mirror orbits on division modulo a prime (W9) lands under Proposition A.1(a). *Two Sparse Binary Operators* [33] is the converse: two three-term rotation-and-shift matrices generate $\mathrm{M}_{32}(K)$ in the optimal word length ten over every field K of characteristic two (Theorem 1 there), and Proposition A.1(d) forces every linear or affine map intertwining both to have rank 0 or 32 (Corollary 5 there), proving that the pair admits only trivial linear contractions. The two twelve-state tensor papers [26, 28] measure

what a composition keeps: the composition tensor C of a twelve-element transformation monoid S satisfies $\operatorname{rank}_+(C) = 35$ and $\operatorname{rank}_+(C^{\otimes q}) = 35^q$ (Theorem 1 of [26], Proposition 7 of [28]), against $18 \le \operatorname{rank}_{\mathbb{R}}(C) \le 20$ and $\operatorname{rank}_{\mathbb{R}}(C^{\otimes q}) \le 20^q$ through the radical quotient $\mathbb{R}[S]/J \cong \mathbb{R}^5 \times \mathrm{M}_2(\mathbb{R})$ (Lemma 5 of [28]).

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